Dynamic coalition formation processes: A generalization of subgame perfection based on socially stable set

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Abstract

We propose two solutions concepts for coalition formation processes derived from socially stable set and applied to dynamic paths: in weakly socially stable set of paths (WSSSP) a coalition is willing to deviate from a path based on a weak preference relation, in conservative socially stable set of paths (CSSSP) a coalition is willing to deviate from a path if all stable continuations following a deviation are at least as good and one is strictly better. WSSSP generalizes the set of perfect equilibrium paths while CSSSP generalizes the set of paths that may be played in subgame perfect Nash equilibrium. The corresponding stable sets - WSSP and CSSP - are related to OSSB and CSSB in the Theory of Social Situations. WSSSP and CSSSP resolve the problem of non existence or emptiness of stable set and stable standard of behavior and allow for the consideration of credible threats which appear in a setting in which coalitions compete for a move.

Keywords Coalition Formation, Stable Set, Farsightedness, Dynamic equilibrium, Theory of Social Situations

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1 Introduction

How should we think about coalition formation processes over time? When considering a deviation from an existing coalition structure, agents may take into account

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discounted future payoffs and they may transition between states with different constraints in terms of available payoffs or possibilities of forming coalitions (see Konishi and Ray [15]). But agents may also raise the consequences of future play in negotiations. Indeed, what underlies the most fundamental solution concepts in cooperative game theory, which attempts to model free negotiations between agents, is either the absence of threats for the outcomes that enter the solution (as in the core) or the balance between threats and counter-threats (as in stable set). Yet, as Ray and Vohra [27] observed,[1] if one were to incorporate credible threats into a solution concept for games with sequential moves one would have to appeal to some form of subgame perfection and, therefore, one would have to borrow from noncooperative game theory.

In this paper we demonstrate that solution concepts derived from socially stable set, a generalization of stable set (Delver and Monsuur [6]), and applied to dynamic paths reproduce properties of subgame perfection in extensive form noncooperative games and also allow for the consideration of credible threats which appear in a setting with overlapping decisive coalitions. In weakly socially stable set of paths (WSSSP) a coalition is willing to deviate from a path based on a weak preference relation, in conservative socially stable set of paths (CSSSP) a coalition is willing to deviate from a path if all stable continuations following a deviation are at least as good and one is strictly better.[2]

When applied to game trees with single player moves - or simple trees - the sets of WSSSP coincide with the set of perfect equilibrium paths (PEP) in the sense of Greenberg [10]. While this set includes paths that are dominated in the conventional (strategic) sense, CSSSP excludes strategically dominated paths and coincides with the set of paths that may be played in subgame perfect Nash equilibrium (SGE).

As socially stable set generally exists it resolves problems of non existence or emptiness of stable set and stable standard of behavior in the Theory of Social Situations. We show that in the absence of decision cycles, the stable sets corresponding to WSSSP and CSSSP - weakly stable set of paths (WSSP) and conservative stable set of paths (CSSP) are closely related to OSSB and CSSB in the theory of social situations. Our approach helps to further clarify the relationship between subgame perfection and stability (see Luo [18] and Luo and Chenghu [16]), and between Optimistic Stable Standard of Behavior (OSSB) and stable set (see Shitovitz [28]): The

[1] Their discussion of example 5.8 of Ray and Vohra [25].

[2] The idea of basing "dominance" on a weak preference relation might seem odd when we follow the interpretation of dominance in cooperative game theory as "blocking". It is closer to the notion of domination in non cooperative game theory and appropriate if the think of a path that dominates as a contemplated possibility, i.e. a path that agents *may* be willing to follow.

union of WSSP includes OSSB - which itself refines subgame perfection. If the dominance relation underlying WSSP establishes in each node a complete order on paths - as is implicit in our definition of a simple tree - its predictions coincide with CSSB and, hence, with PEP, although more generally the union of WSSP refines CSSB.

Recent contributions to coalition formation have emphasised farsightedness: Konishi and Ray [15] introduce equilibrium process of coalition formation (EPCF) as a solution concept for dynamic games, Ray and Vohra [26] introduce coalition sovereignty as a constraint for farsighted stable set and Ray and Vohra [27] address the problem of non-maximizing solutions in farsighted stable set. All these approaches use indirect dominance - as introduced in Harsanyi's [12] critique of myopic stable set - to excludes outcomes from the solution. The underlying assumption is that agents are willing to participate in each step along a path if this path ultimately improves their well-being. Formally, for two nodes a and b, b indirectly dominates a or $b \gg a$, if there exists a sequence of nodes, $a_0, a_1, ...a_m$ with $a_0 = a$ and $a_m = b$, and coalitions $S_0, S_1, ..., S_{m-1}$ with coalition S_j capable of moving from a_j to a_{j+1} and strictly preferring a_m over a_j. Invoking indirect dominance is reasonable because it ensures that only such threats are raised against the status quo outcome which agents are able and willing to carry out. But focusing exclusively on the indirectly dominating sequence may exclude some of the options from which agents would want to choose in a situation where the indirectly dominating sequence is raised as a credible threat against the status quo.[3] In this case, agents may be persuaded into a move that preempts the threat even if the move violates the indirect domination criterion. In the following section we present an example where coalitions can induce alternative paths so that there is an objection against each path. In this case, focusing on the indirectly dominating path is unconvincing.

1.1 Motivating Examples

1.1.1 Customs union formation with a credible threat

The simplest illustration of this problem when coalitions compete for a move is a two-player setting where in some status quo node player A can induce path α with payoff vector $(u_A, u_C) = (2, 0)$ and player C can induce γ with payoff vector $(3, 1)$ while the status quo guarantees $(0, 2)$. In this case, A can threaten C with moving along α but ultimately would like to use this threat to persuade C to move along γ, while C would prefer to stay in the status quo node.

[3] In an atemporal setting this may be considered innocuous because in an open bargaining situation a historical status quo point is generally of less importance. See, however, Pech [21] where coalition formation is conditioned on a political status quo point.

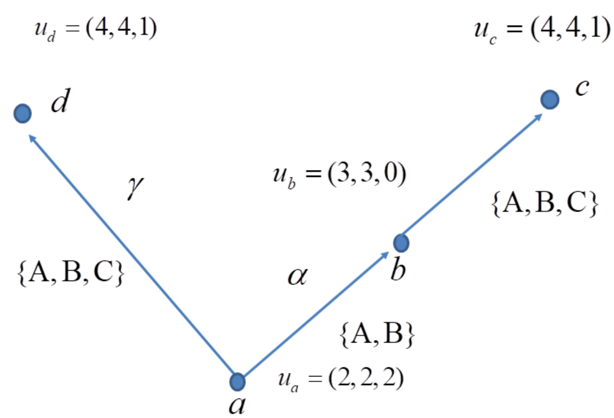

Figure 1: Customs Union Formation Game

To show the economic relevance of this setting, we introduce a customs union formation game inspired by Aghion, Antras and Helpman [1] which is played over two periods and where decisive coalitions overlap. The game is depicted in figure 1.

Example 1. Customs Union Formation The game lasts two periods, each move determines the payoff realization for the current period. In the status quo point a, agents A, B, and C each period realize payoff vector $\{2,2,2\}$. If no coalition moves, agents stay in a. The grand coalition may move along γ to node d where it realizes each period a payoff vector of $(4,4,1)$. Or A and B may move to b where the partition $\{\{A,B\},\{C\}\}$ forms with a payoff vector of $(3,3,0)$. From node b, the grand coalition may move to c with a payoff vector of $(4,4,1)$.

The interpretation is that $\{A,B\}$, by moving along path α, may form a "core" customs union in node b which exerts a negative externality on C, thus attempting to draw C into the grand coalition with lop-sided payoff distribution.

Total undiscounted payoffs along the paths are $\{7,7,1\}$ for path α and $\{8,8,2\}$ for path γ. The status quo point, over two periods, gives a payoff of $\{4,4,4\}$. Once node b is realized, all players gain from continuing on path α to final outcome c. A and B cannot, without C's consent, induce path γ which would give both the highest payoff. α is also inefficient for the grand coalition, when compared to γ.

For path α and path γ there exists a bargaining protocol which implements this path as outcome of a non-cooperative bargaining game: a "closed-loop" bargaining protocol with A or B as proposer where proposals are put against the current status quo in the first and, again, in the second period that can only be accepted or rejected in a single round of bargaining implements path α.[4] Under an "open-loop" rule A proposes the core customs union as a contingent outcome in the first round of bargaining. After B accepts, in the second round A asks the other players to agree on an amendment which offers path γ. C, faced with α as the alternative, has to concede γ.[5]

Most cooperative solution concepts agree that c should be realized if the status quo point is a: letting Z denote the set of nodes, the farsighted stable set $(Z, \gg)$ includes the end points c and d but only c indirectly dominates a along path α. Largest consistent set (LCS, Chwe [5]) agrees with this result: in LCS, a deviation is deterred if the devitiation itself or an indirectly dominating sequence from the deviation results in an outcome in the consistent set which the original deviator does not strictly prefer; again, in point a only the deviation along α to c is predicted. In equilibrium process of coalition formation (EPCF, . Konishi and Ray [15]) players discount utilities over dynamic paths and engage in a deviation from a status quo state if this maximizes their payoff over the dynamic path. In the example, the only coalition which profits from a deviation from the status quo state a is $\{A, B\}$ along path α.

C's preference for the status quo point implies that the bargaining situation is cyclical: C may renege on any agreement to move along γ, thus bringing about its preferred status quo outcome. Because of the cyclicity of the situation, stable standard of behavior (SB) assigns the empty set as solution to node a: in SB, a path is "dominated" if there is a coalition capable of deviating to another path that is included in the solution such that the coalition "prefers" that deviation.[6] In node a, $\{A, B\}$ want to deviate from a by inducing α and $\{A, B, C\}$ want to deviate from α by inducing γ. C wants to deviate from γ and force the default path $\overline{a}$. While identifying the problem, SB does not make a recommendation.

In this paper we define a weak dominance relation on the set of paths:[7] a path

[4] This correponds to the solution proposed by Aghion, Antras and Helpman [1] in an extensive form game of characteristic function bargaining.

[5] See Alimbekov, Madumarov and Pech [2]. Gomes and Jehiel [9] provide a general inefficiency result with a bargaining protocol which is similar to the "closed-loop" variant of the customs union formation example in that it does not allow for contingent agreements.

[6] See Greenberg [10]. Xue [30] applies TOSS to a path situation - our discussion refers to his modelling approach. Section 4 contains a formal definition.

[7] Xue [30] has formulated SB for paths.

α dominates another path β, or $\beta \lhd \alpha$, if the coalition that can replace β with α weakly prefers the outcome that can be reached on α over the outcome that can be reached on β. We can define stable set in node a, $V(a, \lhd)$, for all paths with stable continuations which emanate in node a.[8] Stable set is internally stable, i.e. $\alpha, \beta \in V$ implies that neither $\alpha \lhd \beta$ nor $\beta \lhd \alpha$, and externally stable, i.e. for all $\beta \notin V$ there is $\alpha \in V$ such that $\beta \lhd \alpha$. It is easy to check that because of the cyclicity in node a no set of paths satisfies these conditions, hence stable set does not exist.

Obviously, the bargaining situation in node a is such that without further information on the bargaining process we cannot decide which path agents are going to follow. So a solution which reflects this bargaining situation has to admit all possible outcomes. Formal approaches to resolve this problem for stable set have been based on the transitive closure of the dominance relation (see, e.g., Peris and Subiza [23]) which in our case takes the form $\lhd\!\lhd$: $\alpha_1 \lhd\!\lhd\, \alpha_m$ if and only if there exists a sequence such that $\alpha_1 \lhd \alpha_2, ..., \alpha_i \lhd\ \alpha_{i+1}, ..., \alpha_{m-1} \lhd\ \alpha_m$. Delver and Monsuur [6] restrict the transitive closure relation $\lhd\!\lhd_{|V}$ to elements α_i in the set V. Adapting their definition of socially stable set, we use the relation $\beta \lhd_{|V} \alpha$ which is true iff $\beta \lhd \alpha$ and there is no sequence such that $\alpha \lhd\!\lhd_{|V} \beta$. An appealing property of their solution is that every stable set $V(\lhd)$ is also a socially stable set $V(\lhd_{|V})$. Accordingly, we define the stable set $V(\lhd_{|V}) \subseteq X$ by saying $\beta \in X \backslash V(\lhd_{|V})$ if and only if there is $\alpha \in V$ such that $\beta \lhd_{|V} \alpha$.[9] In the example, we obtain the weakly socially stable set of paths as $V(a, \lhd_{|V(a)}) = \{\overline{a}, \alpha, \gamma\}$.

As we are going to show, our solution concept generalizes subgame perfection for dynamic games of coalition formation. Another solution method which generalizes subgame perfection using backward induction is Kimya's [14] Equilibrium Coalitional Behavior (ECB). The equilibrium is defined as a set of paths from which there is no strictly profitable coalitional deviation. Coalitional deviations may be blocked by a subcoalition of the coalition that can instigate the deviation: α is blocked by γ but γ itself is blocked by $\overline{a}$, so $\overline{a}$ is the unique ECB. Dutta and Vohra [7] take a different approach to derive this result: in their definition of strong rational expectations farsighted stable set (SREFS) they require that if a state is non stationary (i.e. if it is dominated), it must be dominated via an optimally profitable path in the sense that no subcoalition of the initiator benefits by joining another coalition in bringing about a different outcome. In our case, suppose that $\{A, B\}$ initiate α. But $\{A, B\}$ are both better off by joining $\{A, B, C\}$ in bringing about γ. In this case, the maximizing coalition does not form because $\{C\}$ prefers to stay in a.

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[8] In our case, α has a stable continuation $\underline{\alpha}_b$ in b, because $\overline{b} \lhd \underline{\alpha}_b$.

[9] Delver and Monsuur's define socially stable set for a strict dominance relationship. See also remark 1.

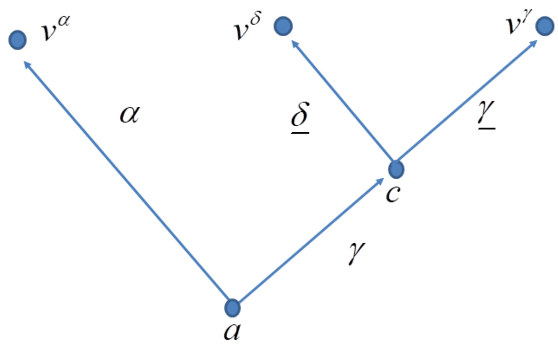

Figure 2: Illustration of examples 2 and 6

1.1.2 Subgame perfection and stability

The following example illustrates the relationship between subgame perfection and weakly stable set in "simple trees" with single player moves. Because in such games WSSSP and CSSSP coincide with the corresponding stable sets, we only need to discuss WSSP and CSSP. Consider the game depicted in figure 2.

Example 2. In the root a, player A chooses between α and γ. In node c, player C chooses between the truncated paths $\underline{\gamma}$ and $\underline{\delta}$. We assume that C is indifferent between the terminal nodes. For player A we consider the preference orders:
(a) $\gamma \succ_A \alpha \succ_A \delta$,
(b) $\gamma \succ_A \alpha \sim_A \delta$.

The subgame perfect equilibria are $\{\alpha, \underline{\delta}\}$ and $\{\gamma\}$ in case (a) and $\{\gamma\}$ for A and $\underline{\gamma}$ or $\underline{\delta}$ for C in case (b). However, $WSSP$ returns $V^1(X, \triangleleft) = \{\alpha, \underline{\delta}\}$ and $V^2(X, \triangleleft) = \{\gamma\}$ in both cases: at stage c, we obtain $V^1(c, \triangleleft) = \{\underline{\delta}\}$ and $V^2(c, \triangleleft) = \{\underline{\gamma}\}$. Fixing $V^1(c, \triangleleft) = \{\underline{\delta}\}$ gives $V^1(X, \triangleleft) = \{\alpha, \underline{\delta}\}$ and fixing $V^2(c, \triangleleft) = \{\underline{\gamma}\}$ gives $V^2(X, \triangleleft) = \{\gamma\}$ for the overall game.

Conservative stable set of paths, in case (a), returns $V(X, \triangleleft^Y) = \{\gamma, \alpha, \underline{\delta}\}$ which coincides with the set of subgame perfect equilibrium paths and the set of weakly

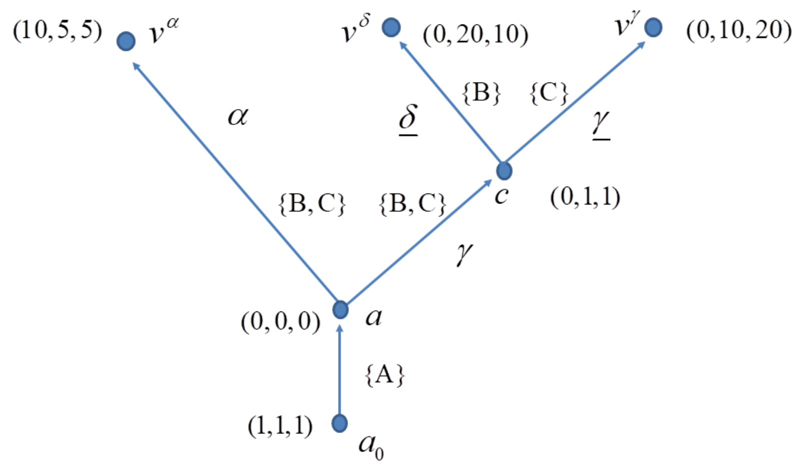

Figure 3: Illustration of example 3 by Ray and Vohra (2014)

stable paths. In case (b), CSSP refines $WSSP$: γ conservatively dominates α as all continuations of γ are at least as good as α with one strictly preferred continuation. Hence, $V(X, \triangleleft^Y) = \{\gamma, \underline{\delta}\}$. As holds more generally, CSSP coincides with the set of subgame perfect equilibrium paths.

1.1.3 Maximality

In farsighted stable set a deviating coalition does not necessarily pick a maximizing deviation.[10] While SB is capable of resolving this problem, it fails to do so when it assigns the empty set at some node. WSSSP, which always exists, overcomes this problem: consider the following example, due to Ray and Vohra [25].

Example 3. Figure 3 expands the graph in figure 2 with node a_0 inserted ahead of node a. In a_0, A has to decide whether to move to a. In a, $\{B, C\}$ decide between α and γ but in node c, each of them can act unilaterally. While B and C prefer both outcomes reached from c to α, A's prefers α but would rather stay in a_0 than realizing any of the outcomes that can be reached from c.

[10]Ray and Vohra [27] refine farsighted stable set to address this problem.

In c, B and C want to move along different paths, hence SB assigns the empty set to node c. Therefore, SB assigns path α as solution to node a_0. But inititating α by moving to a is irrational for player A: while it is unclear which path will be followed at node c either $\underline{\gamma}$ or $\underline{\delta}$ is going to prevail, both of which are worse for A than a_0. On the other hand, (socially) stable set of paths assigns the default path $\overline{a}_0$ to a_0 and socially stable set of paths generally avoid problems of emptiness or non existence of the solution.

Our paper is organized as follows: section 2 sets up our dynamic coalition formation model and introduces WSSSP. Section 2.1 establishes the relationship with subgame perfect equilibrium. Section 3 introduces CSSSP and establishes the relationship with subgame perfection. Section 4 establishes the relationship between SB and the different versions of stable set of paths. Section 5 discusses extensions to unbounded games and cyclic graphs. Section 6 concludes.

2 Weakly Socially Stable Set of Paths (WSSSP)

We consider an n-person game of perfect information without chance moves. Let Z be the set of nodes. X is the set of feasible paths connecting these nodes and $X(v_t)$ the finite set of paths which can be reached from node v_t including v_t itself. Agents have preferences $\succsim_i (v_t)$ defined on $X(v_t)$. Feasibility of paths may reflect institutional or factual relationships between situations over time. Let a and b denote two nodes. An effectivity relation $a \xrightarrow{S} b$ signifies that in situation a, coalition S can bring about situation b. $a \xrightarrow{S} b$ implies that $b \in \Omega(a)$, the set of successor nodes of a. We denominate $\underline{\alpha}_{v_t}$ the truncated subsection of α starting in node v_t. The option of remaining in node a is designated as path $\overline{a}$.

Definition 1. A path $\gamma \in X(a)$, other than the default path $\overline{a}$, starts in node a and consists of a sequence of successive nodes where each node has one predecessor and one successor on that path, except for start and end node of a path. Successive nodes are connected by an effectivity relation $\xrightarrow{S}$. Paths diverge at every non trivial decision node and never merge.

The assumption that paths never merge is slightly more restrictive than the environment in Xue [30]. Essentially, we assume that different histories are represented by different paths as is appropriate for a theory of dynamic coalition formation. The environment considered in this paper is more general than Greenberg's [10] tree situation - which corresponds to standard extensive form games and our simple tree setting - in that it allows for different players or coalitions to move in each node and it allows players to receive a payoff not only in the terminal nodes.

Definition 2. A path $\gamma \in X(a)$ dominates path $\alpha \in X(a)$ via $\triangleleft$, i.e. $\alpha \triangleleft \gamma$, if there is $c \in \gamma$, $c \notin \alpha$, and S such that $a \xrightarrow{S} c$, and $\gamma \succsim_S \alpha$.

γ dominates α if at their junction a there is a coalition S whose members are capable and willing to veer off path α. $\triangleleft$ captures the notion that a weakly dominating path "may" prevail. Agents may prefer to remain in the status quo point rather than moving away from it. The default path $\overline{a}$ is realized if no coalition wants to deviate from a and it dominates another path α in $X(a)$ via $\triangleleft$ if for every coalition S which is effective for α, $\alpha \not\succ_S \overline{a}$.

It is straightforward to define stable set based on the dominance relation $\triangleleft$. However, stable set does not exist when the dominance relationship is cyclic at any node (and the cycle has odd length).[11] As we have argued in the discussion of example 1, an appropriate way of dealing with cycles is to say that any element of the cycle may be selected and to include the cycle in the solution. Including a particular element in the solution may encourage a deviation (as in WSSSP) or discourage it (as in CSSSP). Building on Delver and Monsuur's (2001) definition of socially stable set we say that two elements may be selected in a solution if one element dominates the other and this relationship is "balanced" by the latter element dominating the former via the transitive closure of a sequence of paths which are all included in the stable set. So we define the dominance relation $\triangleleft_{|D}$ conditional on $D \subseteq X$:

Definition 3. $\alpha \triangleleft_{|D} \beta$ iff $\beta \triangleleft \alpha$ and $\nexists\ \gamma_0, \gamma_1, ..., \gamma_K \in D$, $D \subseteq X$, $K \geq 1$ with $\gamma_0 = \alpha$ and $\gamma_K = \beta$ such that $\gamma_0 \triangleleft \gamma_1 \triangleleft ... \gamma_{K-1} \triangleleft \gamma_K$.

Socially stable set, like stable set, is free of contradictions: no path in the stable set dominates another path in the set. And it accounts of all paths: paths not in the set are dominated by paths in the set.

Definition 4. For every node $a \in Z$, the set $V(X(a), \triangleleft_{|V})) \subseteq X(a)$ is weakly internally socially stable if $\alpha \in V$ implies that there is no $\beta \in V$ such that $\alpha \triangleleft_{|V} \beta$. And it is weakly externally socially stable if for all $\gamma \in X(a) \backslash V$ there is $\alpha \in V$ such that $\gamma \triangleleft_{|V} \alpha$. The weakly socially stable set of paths, $V(X, \triangleleft_{|V})$ assigns $V(X(a), \triangleleft_{|V})$ to all all non terminal nodes $a \in Z$ such that $V(X, \triangleleft_{|V})$ is weakly internally and weakly externally socially stable.

[11] A sufficient condition for existence of $V(\triangleleft)$ is that the graph is acyclic. A graph is cyclic at node a if there are $K > 1$ distinct paths $\alpha_k \in X(a)$ and a sequence that satifies $\alpha_k \triangleleft \alpha_{k+1}$, $k = 1, ..., K-1$, $\alpha_K \triangleleft \alpha_1$ and there is no relation $\alpha_{k+1} \triangleleft \alpha_k$ for any k. Acyclicity ensures that there is at least one element which is maximal with respect to $\triangleleft$. Acyclicity of $\triangleleft$ implies but is not implied by acyclicity of the strict dominance relationship $\succ$ (see appendix).

$V(X, \triangleleft_{|V})$ is a weakly socially stable set of paths if for any path α in the set, its continuation at any $a \in \alpha$ is in $V(X(a), \triangleleft_{|V})$. Even when allowing for their strict dominance relation to be replaced by a weak dominance relation, Delver and Monsuur [6] define socially stable set slightly differently, stating that $\alpha, \beta \in V$ if $\alpha \triangleleft\triangleleft_{|V} \beta$ implies $\beta \triangleleft\triangleleft_{|V} \alpha$ and for all $\beta \notin V$ there is $\alpha \in V$ such that $\beta \triangleleft \alpha$. However, as we show in the appendix, their and our definition describe the same set:

Remark 1. Definition 4 is equivalent (up to the weak dominance relation) to Delver and Monsuur's definition of socially stable set.

Establishing existence of WSSSP is straightforward:

Proposition 1. *Weakly socially stable set of paths exists.*

Socially stable set is typically non unique. For example, for $X = \{\alpha, \beta\}$ with the weak dominance relationships $\alpha \triangleleft \beta$, $\beta \triangleleft \alpha$, there are three weakly socially stable sets: $V^1(\triangleleft_{|V}) = \{\alpha\}$, $V^2(\triangleleft_{|V}) = \{\beta\}$, $V^3(\triangleleft_{|V}) = \{\alpha, \beta\}$ rather than two stable sets as is the case with $V(\triangleleft)$.

Note that the union of socially stable sets, $\Sigma V(\triangleleft_{|V})$, is not itself socially stable: assume $\alpha \not\triangleleft \beta$ and $\beta \not\triangleleft \alpha$ but each is dominated by another alternative, i.e. $\alpha \triangleleft \gamma$ and $\beta \triangleleft \delta$ and we also have the pair of relations $\gamma \triangleleft \delta$ and $\delta \triangleleft \gamma$ for which we write $\gamma \leftrightarrow \delta$. The weakly socially stable sets of paths are $V^1(\triangleleft_{|V}) = \{\gamma, \beta\}$, $V^2(\triangleleft_{|V}) = \{\alpha, \delta\}$ and $V^3(\triangleleft_{|V}) = \{\gamma, \delta\}$. Yet the union of socially stable sets is $\{\alpha, \beta, \gamma, \delta\}$ which is not socially stable.

From definition 4 it is immediate that, if $V(X, \triangleleft)$ exists, it is also socially stable: if $\beta \triangleleft \alpha$, it is always possible to exclude β from the socially stable set, thereby severing the cycle back to α. On the other hand, not every path in a socially stable set is contained in the corresponding set $V(\triangleleft)$, even when it exists:

Example 4. Assume the following relationships exist between paths in X: $\alpha \leftrightarrow \beta$ and $\alpha \leftrightarrow \gamma$ but $\gamma \triangleleft \beta$.

As $\alpha \leftrightarrow \beta$, if α or β are included in some $V(\triangleleft)$, only one $\triangleleft$-relation may be active, so either α or β is included in any one stable set. In this case, $V^1(\triangleleft) = \{\alpha\}$ and $V^2(\triangleleft) = \{\beta\}$. Yet $\{\alpha, \beta, \gamma\}$ is a weakly socially stable set $V(\triangleleft_{|V})$.

However, if $\triangleleft$ establishes a complete order on $X(z)$, the predictions of $V(\triangleleft)$ and $V(\triangleleft_{|V})$ coincide:

Proposition 2. *Assume that at each z, $\triangleleft$ establishes a complete order on $X(z)$. Then $\alpha \in \Sigma V(\triangleleft_{|V})$ if and only if $\alpha \in \Sigma V(\triangleleft)$.*

Completeness with transitive preferences ensures that the transitive closure $\triangleleft\triangleleft_V$ collapses to $\triangleleft$. One implication of proposition 2 is that for analyzing simple trees we can focus on $V(\triangleleft)$ and extend the results to weakly socially stable set of paths.

2.1 Relationship between weakly stable set and SGE

Because of proposition 2 it is sufficient to focus on $V(X, \triangleleft)$. Our definition of a game tree with single player moves corresponds to Greenberg's [10] tree situation:

Definition 5. A game is a simple tree if in each node one agent moves, all nodes except possibly the terminal nodes give a payoff of zero and staying in any non terminal node is not a permissible move.

Because agents have standard preferences $\succsim_i$, it follows that in each node z, $\triangleleft$ establishes a complete order on the set paths $X(z)$. As example 2 in the introduction has demonstrated, $V(X, \triangleleft)$ may include paths which correspond to strategies that do not constitute a subgame perfect Nash equilibrium because they are dominated in the conventional (strategic) sense. However, all subgame perfect paths are included in the set of weakly stable set of paths: let $SGE(v_t)$ be the set of all paths originating in v_t which may be played as part of a subgame perfect equilibrium.

Proposition 3. *Assume a game can be represented as a simple tree. The sets of weakly stable sets $\Sigma V(X(v_t)), \triangleleft)$ and $\Sigma V(X(v_t)), \triangleleft\triangleleft_V)$ include the set of subgame perfect paths, $SGE(v_t)$ of the corresponding extensive form game.*

However, as example 2 shows, $\triangleleft$ makes it "too easy" for a path to dominate and, thus, too many paths are supported in *some* weakly stable set. In the following section we provide a stronger notion of domination where a path dominates another path if all stable continuations of the former are weakly - and one is strictly - preferred to the other path. Surprisingly, making the domination relation more demanding results in a refinement of the solution concept: as a consequence, all paths that are weakly dominated in the conventional (strategic) sense are excluded from any stable set.

To establish the one-to-one relationship between subgame perfect equilibrium and WSSP, we say that a strategy profile of an extensive form game assigns actions $s_{v_t^k}$ to all non terminal nodes $v_t^k \in Z$. $\underline{\alpha}_{v_t^k}$ is an equilibrium path corresponding to this strategy profile if the sequence of equilibrium actions starting in v_t^k induce this path. The one-to-one property - which does not extend to weakly socially stable set - follows from the proof of proposition 3:

Corollary. *Assume a game can be represented as a simple tree. For each path in $\alpha \in SGE(v_t)$ there is one corresponding weakly stable set $V(X(v_t)), \triangleleft)$ with unique solution $\alpha \in V(X(v_t)), \triangleleft)$. Moreover, let $\{s_{v_t^k}\}$ be a subgame perfec strategy profile. Then there exists $V(X, \triangleleft)$ with $\underline{\alpha}_{v_{t+j}^k} \in V(X, \triangleleft)$ such that $\underline{\alpha}_{v_{t+j}^k}$ corresponds to the sequence of actions assigned to v_{t+j}^k.*

3 Conservative stable set of paths (CSSP) and its generalization

In this section we define a "conservative" stable set of paths (CSSP) where agents only deviate from a path α to a path γ if they weakly prefer all stable continuations of γ to α with at least one strictly preferred stable continuation.[12] As a first step, we define the conditional dominance relation given a subset Y of all possible continuations of γ:

Definition 6. Let $Y \subseteq X(c)$ be the set of paths emanating in c. $\alpha \lhd^Y \gamma$ iff there is S and $a \xrightarrow{S} c$, $a \in \alpha$, $c \in \gamma$, $c \notin \alpha$ and for all $\delta \in Y$: $\delta \succsim_S \alpha$ and $\exists \delta' \in Y$ with $\delta' \succ_S \alpha$.

Consider example 3 where in node c in figure 3 $\{B\}$ wants to move along $\underline{\delta}$ and $\{C\}$ wants to move along $\underline{\gamma}$. At c, either path may be selected. Correspondingly, $\underline{\gamma}$ and $\underline{\delta}$ each may be stable but stability does not hold for both paths simultaneously. As previous coalitions want to consider all possible paths of future play they need to condition their actions on the union of stable sets at c. Therefore, for any node a and successor node c we define Y the union of stable sets in the remainder of the game starting in c: for $a = v_t$ with $t < T$ let c be a successor node $c \in \Omega(a)$. For $t = T - 1$ define $Y = c$ and for $t < T - 1$, $Y \subseteq X(c) : Y = \cup_k V^k(X(c), \lhd^Y)$. Again, $V(X, \lhd^Y)$ is a stable set if it is internally and externally stable for the dominance relation $\lhd^Y$.

Definition 7. $V(X, \lhd^Y) \subseteq X$ is a conservative stable set of paths if $(X, \{\lhd^Y\})$ is internally stable, i.e. $\alpha \in V(X, \lhd^Y)$ implies that there is no $\alpha \in V(X, \lhd^Y)$ such that $\alpha \lhd^Y \beta$, and it is externally stable, i.e. for all $\gamma \in X \backslash V(X, \lhd^Y)$ there is $\alpha \in V(X, \lhd^Y)$ such that $\gamma \lhd^Y \alpha$.

3.1 Simple trees

In the case where $\left(X, \{\xrightarrow{S}\}, \{\succsim_S\}\right)$ is a simple tree, we can use Luo's [16] generalized stable set to determine CSSP: define a general system $\left(X, \{\succ^A\}_{A \subseteq X}\right)$ where $\alpha \succ^A \beta$ iff $\beta \lhd^A \alpha$ and the generalized stable set $V(X, \succ^A) = \varphi(A)$. In this case there is a unique solution $V(\lhd^Y, X)$ which coincides with the set of subgame perfect (Nash) strategies, as we show section 3.3.

[12] To simplify notation we assume that the coalition agrees on the strictly preferred path(s) but the extension to the case where individual strict preferences in the coalition differ is straightforward.

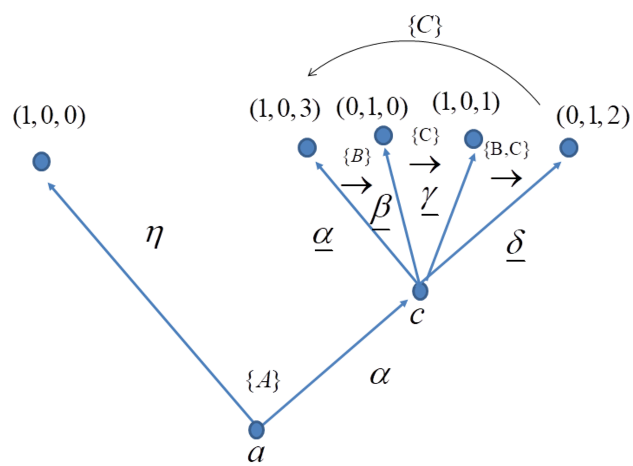

Figure 4: Illustration of example 5

Proposition 4. *Assume that X is a simple tree and that preferences $\{\succsim\}$ define a complete order. Then the game has a unique solution $V(\lhd^Y, X) = \varphi(V)$*

In the case of coalitional moves, the uniqueness result fails to hold as the following example shows:

Example 5. Consider graph 4 with payoff vectors (u_A, u_B, u_C). A moves in the root and in c, B can replace $\underline{\alpha}$ with $\underline{\alpha}$, C can replace $\underline{\beta}$ with $\underline{\gamma}$, and so on.

Stable sets in node c are $V_{11} = \{\underline{\beta}, \underline{\delta}\}$, $V_{12} = \{\underline{\alpha}, \underline{\gamma}\}$, $V_{13} = \{\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}\}$. In node c, we obtain the stable sets $V_0(\succ^{V_{11}}) = \{\eta\}$, $V_0(\succ^{V_{12}}) = \{\eta, \underline{\alpha}, \underline{\gamma}\}$, $V_0(\succ^{V_{13}}) = \{\eta\}$. Clearly, a conservative decision maker would have to cater for the possibility that any of the moves in $X(c) = \{\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}\}$ will be realized and, as $V_0(\succ^{X(c)}) = \{\eta\}$. However, $X(c)$ is not stable.

3.2 The general case: Conservative socially stable set of paths CSSSP)

In the general case with coalitional moves, $V(\lhd^Y, X)$ may be non unique and, if the dominance relation is cyclic in any node, it may not exist. The generalization of the solution method based on socially stable set solves the problem of non existence but multiplicity of equilibria persists. In analogy to definition 3, a path β is dominated by α conditional on some set D if it is dominated via $\lhd^Y$ unless there is a cycle contained in D by which β dominates α via the transitive closure of the dominance relation:

Definition 8. $\alpha \lhd^Y_{|D} \beta$ if $\beta \lhd^Y \alpha$ and $\nexists\, \gamma_0, \gamma_1, ..., \gamma_K \in D$, $D \subseteq X$, $K \geq 1$ with $\gamma_0 = \alpha$ and $\gamma_K = \beta$ such that $\gamma_0 \lhd^Y \gamma_1 \lhd^Y ... \gamma_{K-1} \lhd \gamma^K$.

We define conservative socially stable set of paths, $V(X, \lhd^Y_{|V})$, by restricting the transitive closure relationship to include only paths in the stable set, i.e., $D \equiv V$.

Definition 9. $V(X, \lhd^Y_{|V}) \subseteq X$ is a conservative socially stable set of paths if V is internally stable, i.e. α, $\beta \in V$ if implies $\alpha \not\lhd^Y_{|V} \beta$ and it is externally stable, i.e. $\gamma \in X \backslash V$ implies there is $\alpha \in V$ such that $\gamma \lhd^Y \alpha$.

The observation that the union of all weakly socially stable sets is not socially stable similarly applies to conservative socially stable sets. So there is no conservative socially stable set which necessarily includes all conservative socially stable sets. It follows, that equivalence between (largest) φ-stable set and union of conservative stable sets fails to hold.[13]

Existence of CSSSP follows along the same line as existence of WSSSP: in a tree, a cycle can only emanate over paths that start in the same node, because coalitions need to be able to induce any divergent path. Therefore, example 4 also applies to CSSSP, so $\Sigma V(\lhd^Y)$ need not coincide with $\Sigma V(\lhd^Y_{|V})$ even if the former exists.

3.3 Relationship between conservative stable set and SGE

Consider a game with a representation as a simple tree. Proposition 2 on the equivalence of WSSSP and WSSP in the case of simple trees extends to CSSSP and CSSP so that if $\lhd^Y$ establishes a complete order, so we can focus on CSSP. Example 2 suggests that CSSP resolves the problem of inclusion in WSSP of paths which are dominated in the strategic sense. We can show that, more generally, a path is included in CSSP if and only if it is in SGE.

Proposition 5. *Consider a game with a representation as a simple tree: there is $V(X, \{\lhd^Y\})$ such that path $\alpha \in V(X, \{\lhd^Y\})$ if and only if $\alpha \in SGE$.*

The union of all paths in CSSP, $\Sigma V(X, \lhd^Y) = \cup_j V^j(X, \lhd^Y)$, collects all paths in SGE which may be played as part of a subgame perfect Nash equilibrium. It is straightforward to show that there is no one-to-one relationship between subgame perfect equilibrium and conservative stable set.

[13] Unlike CSSSP, CSSB can be represented as a generalized stable set, see Luo [17].

4 Relationship between weakly stable set, conservative stable set and stable SB

Our solution concept is closely related to the Theory of Social Situations (TOSS) as introduced by Greenberg [10] and applied to path situations by Xue [30] and Greenberg, Monderer, Shitovitz [11]. TOSS distinguishes between optimistic stable standards of behavior (OSSB) and conservative stable standard of behavior (CSSB). A path is an OSSB (CSSB) if there is no deviation from the path such that *some* (*all*) possible continuations which are themselves in OSSB (CSSB) are strictly preferred by the deviating coalition (Xue, [30]):

Definition 10. σ is an OSSB (CSSB) if for all $a \in Z$, $\alpha \in X(a) \backslash \sigma(a) \Leftrightarrow$ there exist $S \subset N$, $b \in \alpha$, and $c \in Z$ such that $b \xrightarrow{S} c$, $(\sigma(c) \neq \emptyset)$ and $\alpha \succ_S \beta$ for *some* (*all*) $\beta \in \sigma(c)$.

In example 2 (a), the unique OSSB is $\{\gamma\}$ in a and $\{\underline{\gamma}, \underline{\delta}\}$ in node c and the unique CSSB is $\{\alpha, \gamma, \underline{\delta}\}$. So here OSSB and CSSB coincide with WSSP and CSSP. In example 2 (b), WSSP, CSSB and OSSB still obtain the same solution as in the case of example 2 (a), but CSSP agrees with SGE and refines this solution to return $\{\gamma, \underline{\delta}\}$.

If the solution is empty or multi-valued at any node, the solution concepts further differ. Consider the following variant of example 2:

Example 6. S_a has the strict preference order of example 2 (a), i.e. $\gamma \succ_{S_a} \alpha \succ_{S_a} \delta$, but at node c coalition S_γ can induce v^γ and coalition S_δ can induce v^δ. Both prefer either terminal node to staying in c but in preference order (a) each prefers the terminal node that they themselves can induce while in preference order (b) each prefers the terminal node that the other coalition can induce:
(a) $\gamma \succ_{S_\gamma} \delta$ and $\delta \succ_{S_\delta} \gamma$
(b) $\delta \succ_{S_\gamma} \gamma$ and $\gamma \succ_{S_\delta} \delta$

In case (a), $\sigma^{SB}(c) = \emptyset$: both coalitions want to deviate from the status quo node and from the path that the other coalition prefers. In c, SB does not make any prediction - even though the situation changes little from the case of example 2: the unique SB of this game has $\alpha \in \sigma^{SB}(a)$. WSSP and CSSP non uniquely assign a value to node c as $\underline{\gamma} \triangleleft \underline{\delta}$ and $\underline{\delta} \triangleleft \underline{\gamma}$: $V^1(c, \triangleleft^Y) = V^1(c, \triangleleft) = \{\underline{\gamma}\}$ and $V^2(c, \triangleleft^Y) = V^2(c, \triangleleft) = \{\underline{\delta}\}$. For the overall game, $V^1(X, \triangleleft) = \{\gamma\}$ and $V^2(\bar{X}, \triangleleft) = \{\alpha, \underline{\delta}\}$ which corresponds to the prediction of subgame perfection in the closely related case of example 2 while CSSP similarly returns $\{\alpha, \gamma, \underline{\delta}\}$. Socially stable set additionally admits $V^3(c, \triangleleft^Y_{|V}) = V^3(c, \triangleleft_{|V}) = \{\underline{\gamma}, \underline{\delta}\}$, in which SSSP optimistically selects γ (but also supports $\alpha, \underline{\delta}\}$ while CSSSP conservatively does not exclude anything.

In case (b), $\sigma^{SB}(c) = V(c, \triangleleft) = \{\underline{\gamma}, \underline{\delta}\}$: while no coalition wants to stay in c, none would want to deviate from the path that the other coalition induces. For stable set of paths, the dominance relationships are $\underline{\gamma} \ntriangleleft \underline{\delta}$ and $\underline{\delta} \ntriangleleft \underline{\gamma}$. In node a, S_a may decide optimistically or conservatively, so $\{\alpha, \gamma\} \in \sigma^{CSSB}(a)$ and $\gamma \in \sigma^{OSSB}(a)$. At node c, stable set is also multi-valued. In this case, $V(X, \triangleleft)$ optimistically selects γ while $V(X, \triangleleft^Y)$ does not exclude anything.

A first observation from example 6 (b) concerns the relationship of WSSP and CSSP: in the case where the solution is multivalued at some node, there may paths in $\Sigma V(\triangleleft^Y)$ which are not in $\Sigma V(\triangleleft)$ as CSSP blocks conservatively while WSSP blocks optimistically. As we have shown in the previous section, in simple trees CSSP refines WSSP. The following propositions summarizes the consequences of these two observations:

Proposition 6. *Assume that $V(\triangleleft)$ and $V(\triangleleft^Y)$ exist. CSSP neither includes nor is it included in WSSP, i.e. $\Sigma V(X, \triangleleft^Y) \nsubseteq \Sigma V(X, \triangleleft)$ and $\Sigma V(X, \triangleleft) \nsubseteq \Sigma V(X, \triangleleft^Y)$.*

Example 6 (a) shows that cyclicity of the dominance relation differently affects stable set and SB. Example 6 (b) shows that the similarity of predictions of CSSB and WSSP observed in part (a) critically depends on completeness of order.

In the following we focus on the case where SB and $V(\triangleleft)$ or $V(\triangleleft^Y)$ exist and are non empty. Note that existence of $V(\triangleleft)$ does not imply existence of OSSB. [14]

Proposition 7. *Assume that $V(X, \triangleleft)$ and OSSB exist and that σ^{OSSB} is non empty at every node. Then $\sigma^{OSSB} \subseteq \Sigma V(\triangleleft)$ but the reverse is generally not true.*

To show that the reverse is not generally true consider example 2:[15] The unique OSSB of this game is $\sigma^{OSSB} = \gamma$ which is also an WSSP: S_a expects player C to choose the favorable path $\underline{\gamma}_{|c}$. There is another WSSP: $V(\triangleleft = \underline{\delta}_{|c}, \alpha)$. Hence, $\sigma^{OSSB} \nsupseteq \Sigma V(X, \triangleleft)$.

There is a close relationship between stable set of paths (WSSP) and conservative stable standard of behavior (CSSB): if $\{\triangleleft\}$ establishes a complete order on the paths emanating in any node and the effectivity relationship is symmetric, i.e. if $\alpha \overset{S}{\to} \beta$ implies $\beta \overset{S}{\to} \alpha$, the two concepts coincide. Simple trees satisfy both conditions. On the other hand, in the general case neither the set of $CSSB$ nor $WSSP$ includes the other.

[14] Assume that the dominance relations in example 4 are strict. In this case, OSSB does not exist although stable set of paths exists.

[15] See also Tadelis [29], who shows that OSSB refines subgame perfection and, hence, CSSB.

Proposition 8. *Assume that $V(X, \lhd)$ and CSSB exist and σ^{CSSB} is non empty at all $z \in Z$. Moreover, assume that for all nodes $z \in Z$, $\lhd$ establishes a complete order on $X(v)$ and the effectivity relation $\{\rightarrow\}$ is symmetric, i.e. that coalition S can replace β with α implies that S can replace α with β. Then $\sigma^{CSSB} = \Sigma V(X, \lhd)$. If and the $\Sigma V(X, \lhd) \subseteq \sigma^{CSSB}$. If completeness and symmetry do not hold, neither $\Sigma V(X, \lhd) \subseteq \sigma^{CSSB}$ nor $\sigma^{CSSB} \subseteq \Sigma V(X, \lhd)$.*

This proposition explains why, although OSSB is closely related to stable set (Shitovitz [28]) and, as Luo [18] and this paper show, stability is related to subgame perfection, it is CSSB that coincides with the set of perfect equilibrium paths. Symmetry and completeness ensure that the induced preferences on paths of the coalitional game are in accordance with preferences on paths in a simple tree. In this sense, proposition 8 does not appreciably extend the result of proposition 3 for the case of simple trees. Claim 2 in the proof of proposition 8 establishes that $\Sigma V(X, \lhd) \not\subseteq \sigma^{CSSB}$.

Example 6 with preference order (b) demonstrates that $\sigma^{CSSB} \not\subseteq \Sigma V(X, \lhd)$. Note the role of completeness: coalitions moving in node c, S_δ and S_γ, each prefer the end node that the other coalition can induce and, hence, $\lhd$ is not complete: at c we have neither $\underline{\delta} \lhd \underline{\gamma}$ nor $\underline{\gamma} \lhd \underline{\delta}$. In this example, $\delta \in \sigma^{CSSB}(c)$ and $\delta \not\succ \alpha$, hence $\alpha \in \sigma^{CSSB}(a)$. Moreover, $\underline{\gamma}$, $\underline{\delta} \in V(X(c), \lhd)$, but because $\alpha \lhd \gamma$ and γ cannot be excluded from any weakly stable set, α is not stable. In applying stable set, there is the implicit assumption that agents are optimistic in their assessment of continuation of play, hence γ is an effective objection to α. CSSB, by contrast, assumes that agents are conservative in assessing continuation of play, hence it is sufficient that possible outcome δ is not preferred to α, to deter a deviation from α. Now change the example such that a single player C with a complete preference order moves in node c. In this case, γ and δ are only in σ^{CSSB} if C is indifferent which translates into $\underline{\delta} \lhd \underline{\gamma}$ and $\underline{\gamma} \lhd \underline{\delta}$. This is sufficient to ensure that α is included in some weakly stable set.

This result does not extend to weakly socially stable set: consider the case where for a decisive coalition $\gamma \succ \alpha$ and there are decisive coalitions with $\alpha \sim \beta$ and $\beta \sim \gamma$. Clearly, α is included in socially stable set but not in CSSB.

Finally, we relate CSSP to SB. While CSSP and OSSB may overlap, neither set generally includes the other.

Proposition 9. *Assume that $V(X, \lhd^Y)$ and OSSB exist and σ^{OSSB} is non-empty at all $z \in Z$. Then CSSP neither includes nor is included in OSSB.*

5 Extensions

5.1 Unbounded games

Consider the example of an infinite game due to Greenberg [10]: players one and two move sequentially with player one calling u or d and player two calling l or r. After an infinite sequence where u, d, l and r each have been called infinitely often, player one receives 5 and player two receives 0. After a sequence where at least one of the choices u, d, l or r have only been called finitely often, player one receives zero and player two receives 5. Player two can ensure 5 by choosing only l and never r whenever it is her turn, so SGE predicts a payoff of $(0, 5)$. There are, however, two CSSB, one where players receive $(0, 5)$ and one where players receive $(5, 0)$.

Weakly stable set of paths $V(X, \triangleleft)$ for this situation agrees with SGE and excludes paths with infinite sequences u, d, l and r: Consider a node where agent one moves. Path γ starts with picking l and r is never picked. Path β starts with picking r and along β all choices are picked infinitely often. Then $\beta \triangleleft \gamma$ and $\gamma \ntriangleleft \beta$.

Using conservative stable set of paths, $V(X, \triangleleft^Y)$ and applying definition 6, $\gamma \ntriangleleft^Y \beta$ because not all possible paths after an initial move in the direction of β are at least as good as γ. On the other hand, $\beta \triangleleft^Y \gamma$ because all possible paths after an initial move in the direction of γ are at least as good as β.

5.2 Cyclic Graphs

Graphs of dynamic games, as considered in this paper, are not cyclic. However, we may apply the definition of stable set to an environment that permits cyclic graphs. Consider the cyclic graph in figure 5. WSSSP and CSSSP assign solutions in both cases. For comparison with SB, we focus on WSSP and CSSP. Note that OSSB does not exist but CSSB does.[16]

For the graph in figure 5, we cannot construct WSSP. The argument parallels the demonstration of non existence of OSSB: suppose that $\overline{v}_2 \in V(v_2, \triangleleft)$, hence, $\beta \in V(v_1, \triangleleft)$. But then $\alpha \notin V(v_3, \triangleleft)$ by internal stability and $\gamma \in V(v_2, \triangleleft)$ by external stability. From this follows $\overline{v}_2 \triangleleft \gamma$ contradicting that $\overline{v}_2 \in V(v_2, \triangleleft)$. Note that an appropriately defined socially stable set exists.

On the other hand, CSSP, like CSSB, exists: assume that γ and $\overline{v}_2 \in Y(v_2)$. Then $\overline{v}_1 \ntriangleleft^Y \beta$ because not all $\delta \in Y(v_2)$ satisfy $\delta \succsim_{\{B,C\}} \overline{v}_1$. The same construction can be repeated for all nodes, so that the unique CSSP consists of the nodes $\{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$.

[16]See Pech [22] on sufficient conditions for existence of OSSB in cyclic graphs.

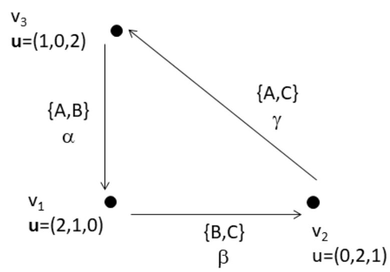

Figure 5: Cyclic Graph

6 Conclusion

In this paper we have two proposed versions of socially stable set (Delver and Monsuur, [6]) applied to paths as solution concepts for dynamic coalitional games which can be represented as trees. In weakly socially stable set of paths (WSSSP), a path is dominated by another path, if an effective coalition weakly prefers the other path and there is no sequence of weakly dominating paths - corresponding to the transitive closure - which is fully included in the stable set and arcs back from the path to the other path. Conservative socially stable set of paths (CSSSP) is similarly constructed but for a path to be dominated by another, an effective coalition weakly prefers all (stable) continuations of the other path and strictly prefer one.

Socially stable set of paths always exists and, hence, addresses shortcomings of stable set and stable standard of behavior (SB). We show that the stable set corresponding to WSSSP and CSSSP - WSSP and CSSP - are closely related to SB.

In game trees with singleton players ("simple trees") we can focus on WSSP and CSSP because they exists and every every stable set is also socially stable. Each path in WSSP coincides with a perfect equilibrium paths while CSSP coincides with the set of paths which may be played in subgame perfect Nash equilibrium. In this sense, WSSSP and CSSSP generalize subgame perfection. Compared to solution concepts for cooperative games which are not based on stability, socially stable set of paths also includes such paths in the solution which are not indirectly dominating the status quo point but which agents might be persuaded to agree to because some (sub)coalition has a credible threat against the status quo.

7 Appendix: Proofs

Proof of remark 1   Define $Q(\alpha,\beta)$ the set of all paths in a sequence: $\alpha \lhd\lhd \beta$, $\beta \lhd\lhd \alpha$ and let $\widetilde{V}(\lhd\lhd_{|V}) \subseteq X$ be a socially stable set as defined by Delver and Monsuur: for all $\gamma \notin \widetilde{V}(\lhd\lhd_{|V})$, there is $\alpha \in \widetilde{V}(\lhd\lhd_{|V})$ with $\gamma \lhd \alpha$ and for all paths α, β in $\widetilde{V}(\lhd\lhd_{|V})$, there are no asymmetric dominance relations or $\alpha \lhd\lhd_{|V} \beta$ implies $\beta \lhd\lhd_{|V} \alpha$.

Assume that $D \subseteq X$ is a socially stable set, i.e. $D \equiv \widetilde{V}(\lhd\lhd_{|V})$

$\Leftrightarrow$ [$\alpha \in D$ implies: if $\alpha(\lhd\lhd_{|D})\beta$, then there exists $Q(\alpha,\beta) \subseteq D$]$\wedge$ [$\gamma \notin D$ implies: $\exists \alpha \in D$ such that $\gamma \lhd \alpha$]

$\Leftrightarrow$ [$\alpha \in D$ implies: if $\alpha \lhd \delta$ then there exists $Q(\alpha,\delta) \subseteq D$]$\wedge$ [$\gamma \notin D$ implies: $\exists \alpha \in D$ such that $\gamma \lhd \alpha$ and $\nexists Q(\alpha,\gamma) \subseteq D$]

$\Leftrightarrow$ [$\alpha \in D$ implies: if $\alpha \lhd \delta$ then $\delta \lhd\lhd_{|D} \alpha$]$\wedge$ [$\gamma \notin D$ implies: $\exists \alpha \in D$ such that $\gamma \lhd \alpha$ and not: $\alpha \lhd\lhd_{|D} \gamma$]

$\Leftrightarrow D \equiv V(\lhd_{|V})$

Proof of the claim in footnote 11   Let $\alpha \succ_{\vec{S}} \beta$ signify that a coalition S that can replace β instead of α and strictly prefers α over β. Then $V(\succ_{\vec{S}})$ is a von Neumann-Morgenstern set and $\succ_{\vec{S}}$ is acyclic if there are no cycles of the form $\alpha_{k+1} \succ_{\vec{S}_{k+1}} \alpha_k$, $k = 1,...,K-1$ and $\alpha_1 \succ_{\vec{S}_1} \alpha_K$. Acyclicity of $\succ_{\vec{S}}$ implies acyclicity of $\lhd$: $\succ_{\vec{S}}$-cycle and $\lhd$-cycle are broken by a $\sim$-relation for the effective coalition, including the case of definition 2 (2) for $\lhd$. But acyclicity of $\lhd$ does not imply acyclitiy of $\succ_{\vec{S}}$: a sequence such as $\alpha \lhd \beta \lhd \gamma \lhd \alpha$ with effective coalitions T, S and R is acyclic if it is checked by some reverse relationship such as $\gamma \lhd \beta$ with any effective coalition. But, irrespective of the reverse relationship, it is compatible with there being a cycle of the form $\gamma \succ_{\vec{S}} \beta \succ_{\vec{T}} \alpha \succ_{\vec{R}} \gamma$.

Proof of proposition 1   Following Delver and Monsuur, define $Q^0(\lhd_{|Q^0})$ as the set of top elements with regard to $\lhd$, i.e. a set without "incoming arcs" where an "incoming arc" stands for the relationship "is dominated by an outside element"[17] Clearly, there is such $Q(\lhd_{|Q^0})$: either $\lhd$ has a maximal element or there is a cycle involving elements of a $\lhd$-chain.

Let X^0 be the set of paths starting node in v_{T-1} at the last stage of the game and let $\mathrm{dom}(Q^0) = \{\alpha \in X^0 : \alpha \lhd \beta, \beta \in Q^0\}$ and $X^1 = X^0 \backslash (Q^0 \cup \mathrm{dom}(\widehat{Q}^0))$.[18]

If $X^1 = \emptyset$ we are done. Otherwise, let Q^1 be the set of top elements with respect to $\lhd$ in X^1.

Note that $Q^0 \cup Q^1$ satisfies the internal stability criterion: suppose $\beta \in Q^0$, $\gamma \in Q^0$ and $\beta \lhd_{|(Q^0 \cup Q^1} \gamma$.

Then either (a): there is $Q(\beta, \gamma)$, such that $\beta, \gamma \in Q$, implying either $\gamma \in Q^0$ or $\gamma \in \mathrm{dom}(\beta)$, thus contradicting that $\gamma \in X^1$.

Or (b): there is no $Q(\beta, \gamma)$, such that $\beta, \gamma \in Q$. But then $\beta \lhd \gamma$, contradicting that β is a top-element.

Construct $X^k, k = 2$ and so on. Within a finite number of steps, $X^K = \emptyset$. Define $V = \cup_{k=0,\dots,K} Q^k$. By construction, V is internally stable and $\mathrm{dom}(V) = \cup_{k=0,\dots,K} \mathrm{dom}(Q^k)$ with $V \cup \mathrm{dom}(V) = X^0$. Hence, V is a stable set.

Next consider node v'_{t-1} and assume that for all successor nodes $v_t^k \in \Omega(v_{t-1})$ we have assigned $V(v_t^k)$ and let $\alpha(v_t) \in V(v_t^k)$. By external stability, $\alpha(v'_{t-1}) \in V(v'_{t-1})$ unless there is $\gamma(v'_{t-1}) \in V(v'_{t-1})$ and $\alpha(v'_{t-1}) \lhd_{|V} \gamma(v'_{t-1})$. As we have demonstrated, V exists for $t = T-1$, by induction it exists for all $t < T-1$.

Proof of proposition 2 That every stable set is also socially stable follows from internal stability of stable set which rules out a dominating sequence $\gamma_0 \lhd, \dots, \lhd \gamma_K$ on V as in definition 3. So it suffices to show that in the case of a simple tree, every path in socially stable set is also included in some stable set.

For strict $\lhd$, completenes of order implies that $\beta \lhd_{|V} \alpha$ if and only if $\beta \lhd \alpha$. Assume that $\alpha, \beta \in V(\lhd_V)$. So consider the case where the relation $\lhd$ is weak. So consider the symmetric relation. By the internal stability condition of socially stable set, $\alpha, \beta \in V(\lhd_{|V})$ if $\alpha \lhd \beta$ implies $\beta \lhd\lhd_{|V} \alpha$. By completeness of order, $\alpha \lhd \beta$ implies $\beta \lhd\lhd_{|V} \alpha$ if and only if $\alpha \lhd \beta$ implies $\beta \lhd \alpha$. So assume that $\alpha \leftrightarrow \beta$ and α, $\beta \in V(\lhd\lhd_{|V})$. We have to show, that stable sets exist with $\alpha \in V(\lhd)$ and $\beta \in V'(\lhd)$.

As $\alpha \notin V'$, suppose $\delta \in dom(\alpha)$ dominates β. By completeness of order and

[17] If elements in any given cylce are dominated, mark the cycle as "previously crossed" and move to a cylce at a higher level. Because X is finite, either there is a highest level cycle or $\lhd$ connects to a previously crossed cycle in which case all elements between the higher-level and the previously crossed cycle are included in $Q^0(\lhd_{|Q^0})$.

[18] If there are no cycles, we can always construct a $V(\lhd)$ free of contradiction by selecting $\beta \in Q^0$ and letting $\widehat{Q}^0 = \{\beta \in Q^0 : \beta \not\lhd \alpha, \alpha \in \widehat{Q}^0\}$.

$\alpha \leftrightarrow \beta$, $\delta \lhd \alpha$ and $\beta \lhd \delta$ imply $\beta \leftrightarrow \delta$. So β is a maximal element in $X \backslash \alpha$ and we can construct $\beta \in V'(\lhd)$.

Proof of proposition 3 We have to show that $\alpha \in SGE(v_t)$ implies that there is $V(X(v_t), \lhd)$ such that $\alpha \in V(X(v_t), \lhd)$:

At the last stage, $T-1$, let $\alpha \in SGE(X(v_{T-1}))$ and $\succsim_{T-1}$ be the complete preference order of the decision maker in $T-1$

$\Leftrightarrow \alpha \succsim_{T-1} \beta, \forall \beta \in X(v_{T-1})$

$\Leftrightarrow \beta \lhd \alpha, \forall \beta \in X(v_{T-1})$

$\Leftrightarrow$ There is $V^k(v_{T-1})$ with $\alpha \in V(v_{T-1})$.

Now consider any previous stage $t-1$ and node v_{t-1} and assume that for all successor nodes b, $\underline{\beta}_b \in SGE(b)$ implies that there is $V(b, \lhd)$ with $\beta \in V(b, \lhd)$.

Let $\alpha \in SGE(v_{t-1})$, $a_t \in \alpha$ and $a_t \in \Omega(v_{t-1})$.

$\Rightarrow$ [For all successors $c_t \in \Omega(v_{t-1}) \backslash \{a_t\}$: there exists $\underline{\gamma}_{c_t} \in SGE(c_t)$ such that $\underline{\alpha}_{a_t} \succsim_{t-1} \underline{\gamma}_{c_t}$].

$\Leftrightarrow$ [For all $c_t \in \Omega(v_{t-1} \backslash \{a_t\})$ and $\gamma \in X(v_{t-1})$ with $c_t \in \gamma$: there is at least one path γ' with continuation $\underline{\gamma}'_{c_t} \in \Sigma V(c_t, \lhd)$ such that $\gamma' \lhd \alpha$].

$\Leftrightarrow$ There is $V(v_{t-1}, \lhd)$ with $\alpha \in V(v_{t-1}, \lhd)$. This establishes the proposition.

Note that from the last two steps it follows that if $\gamma \neq \alpha$ and $\alpha, \gamma \in SEG(v_{t-1})$ then $\alpha \lhd \gamma$ and $\gamma \lhd \alpha$ and, hence α and γ are not in the same stable set.

A subgame perfect strategy profile consists of actions $s_{v_t^k}$ at node v_t^k such that at each node and each alternative action $s'_{v_t^k}$ that induces an equilibrium path $\underline{\gamma}_{v_{t+1}^j}$ starting in a successor of v_t^k, $s_{v_t^k} \succsim_t s'_{v_t^k}$. Therefore, if $\underline{\gamma}'_{c_t}$ and α are part of a subgame perfect strategy profile, then $\underline{\alpha}_{a_t} \succsim_{t-1} \underline{\gamma}'_{c_t}$ and α, $\underline{\gamma}'_{c_t} \in V(X, \lhd)$. Suppose that $\underline{\gamma}'_{c_t} \succ_{t-1} \underline{\alpha}_{a_t}$. Then α and $\underline{\gamma}'_{c_t}$ are not part of a subgame perfect strategy profile and $\alpha \lhd \gamma'$, $\gamma' \not\lhd \alpha$, contradicting that $\alpha \in V(X, \lhd)$.

Proof of proposition 4 First note that any $V(\lhd^Y)$ is unique because non uniqueness implies that there is $\alpha \in V^1$, $\beta \in X \backslash V^1$, and $\beta \in V^2$, $\alpha \in X \backslash V^2$. But then we must have $\beta \lhd^Y \alpha$ and $\alpha \lhd^Y \beta$, implying that for one player β strategically dominates α and vice versa, contradicting that preferences are a complete order.

$\gamma \in X(z) \backslash V(z, \lhd^Y)$ iff $\exists a \in \Omega(z)$ and $\alpha \in V(z, \lhd^Y)$ such that $\gamma \lhd^{V(a, \lhd^Y)} \alpha$

$\Leftrightarrow \gamma \in X(z) \backslash V(z, \lhd^Y)$ iff $\exists \alpha \in V(z, \lhd^Y)$ such that $\alpha \succ^{V(a, \lhd^Y)}$

$\Leftrightarrow \gamma \in X(z) \backslash V(z, \lhd^Y)$ iff $\exists \alpha \in V(z, \lhd^Y)$ such that $\alpha \succ^{V(z, \lhd^Y)} \gamma$

$\Leftrightarrow \gamma \in X(z) \backslash V(z, \lhd^Y)$ iff $\exists \alpha \in X(z)$ such that $\alpha \succ^{V(z, \lhd^Y)} \gamma$

$\Leftrightarrow V(z, \lhd^Y) = \varphi(V(z))$.

$\Leftrightarrow V(\lhd^Y)$ is a φ-stable set for the associated general system $(X, \{\succ^A\}_{A \subseteq X})$.

where the step from $\succ^{V(a,\succ^Y)}$ to $\succ^{V(z,\lhd^Y)}$ follows because $V(a,\lhd^Y) \subset V(z,\lhd^Y)$ and paths in $V(z,\lhd^Y) \backslash V(a,\lhd^Y))$ do not dominate any path in z via $\lhd^{V(a,\lhd^Y)}$.

Proof of proposition 5 Show that $V(\lhd^Y)$ with $\alpha \in V(\lhd^Y)$ if and only if $\alpha \in SGE$. Assume that the result has been established for all subgames starting in $c \in \Omega(a)$.

Assume $\alpha \in SGE(X(a))$

$\Leftrightarrow$ [$\forall c \in \Omega(a), c \notin \alpha$: α is a best response for one equilibrium path in $Y(c)$

$\Leftrightarrow$ either:

(a) $[\alpha \succ_A \inf(\delta \in SGE(c)|\{\succsim_A\})]$,

or (b) [$\alpha \sim_A \inf(\delta \in SGE(c)|\{\succsim_A\})$ and α is not strategically dominated]

where (b) requires that either:

(ba) not all equilibrium continuations of γ with $c \in \gamma$ are weakly preferred to the equilibrium continuations of α, i.e. either there exist $\beta \in SGE(b)$ with $\beta \succ_A \alpha$, or

(bb) A is indifferent between all equilibrium continuations of γ and α, i.e. $\inf(\delta \in SGE(c)|\{\succsim_A\}) = \sup(\delta \in SGE(c)|\{\succsim_A\})$ and $\inf(\beta \in SGE(b)|\{\succsim_A\}) = \sup(\beta \in SGE(b)|\{\succsim_A\})$

$\Leftrightarrow \alpha \not\lhd^Y \gamma$ for all $\gamma \in SGE(c)$

$\Leftrightarrow \alpha \in V(a,\lhd^Y)$.

Proof of proposition 6 As we have shown, in the case of simple trees CSSP refines WSSP.

For a case where $\alpha \in \Sigma V(\lhd^Y)$ does not imply $\alpha \in \Sigma V(\lhd)$, consider example 6: V is multivalued at node c with $V(c,\lhd) = V(c,\lhd^Y) = \{\gamma,\delta\}$. α is excluded in $V(\lhd)$ because it is (optimistically) dominated by γ but α is included in $V(\lhd^Y)$ because it is not conservatively dominated by γ and δ.

Proof of proposition 7 Let $\alpha \succ_{\vec{S}} \beta$ signify that a coalition S that can replace β with α and strictly prefers α over β.

1) Show that $\alpha \in \sigma^{OSSB}$ implies $\alpha \in \Sigma V(\lhd)$.

$\alpha \in \sigma^{OSSB}(v_{t-1}) \Rightarrow$ [There is no $\beta \in X(v_{t-1})$ and effective coalition S_{t-1} in v_{t-1} with $\underline{\beta}_{|v_t} \in \sigma(v_t)$ such that $\beta \succ_{S_{t-1}} \alpha$ and $\sigma(v_t) \neq \emptyset$] $\Rightarrow$ [$\forall\beta$ either: (a) $\alpha \sim_{\vec{S}_{t-1}} \beta$ or (b) $\alpha \succ_{\vec{S}_{t-1}} \beta$ or (c) α and β are uncomparable] $\Leftrightarrow \forall\beta$, $\beta \lhd \alpha \vee \alpha \not\lhd \beta$, hence α is maximal with respect to $\lhd$ and $\alpha \in \Sigma V(v_{t-1},\lhd)$.

In the next step, assume that for all $s \geq t-1$: $\sigma^{OSSB}(v_s) \subseteq \Sigma V(v_s,\lhd)$. Consider a predecessor node v_{t-2} and suppose there is π such that $\pi \in \sigma^{OSSB}(v_{t-2})$ but $\pi \notin V(v_{t-2},\lhd)$.

First, assume that $\pi \lhd \delta$ for $\delta \in \sigma^{OSSB}$, $\delta \in V(v_{t-1}, \lhd)$ such that $\pi \notin V(v_{t-1}, \lhd)$. Therefore, $\delta \ntriangleleft \pi$ and $\delta \succ_{\vec{S}_{t-1}} \pi$. Hence, $\pi \notin \sigma^{OSSB}$, contradicting the supposition.

So assume instead that $\delta \notin \sigma^{OSSB}$, $\delta \in V(v_{t-1}, \lhd)$:

$\delta \notin \sigma^{OSSB}(v_{t-1}) \Leftrightarrow$ [there is γ and effective coalition S_t such that $\gamma \succ_{\vec{S}_t} \delta$].

Because $\delta \in V(v_{t-1}, \lhd)$, either $\gamma \lhd \delta$ or there is $\eta \in V$ such that $\gamma \lhd \eta$, $\delta \ntriangleleft \eta$.

In both cases, there is $V(v_{t-2}, \lhd)$ such that $\pi \in V(v_{t-2}, \lhd)$, contradicting the supposition.

Hence, $\sigma^{OSSB}(v_s) \subseteq \Sigma V(v_s, \lhd)$ for $s \geq t-1$ implies $\sigma^{OSSB}(v_{t-2}) \subseteq \Sigma V(v_{t-2}, \lhd)$.

The relationship $\sigma^{OSSB}(v_t) \subseteq \Sigma V(v_t, \lhd)$ clearly holds for a terminal node, i.e. for $v_t = v_T$, for which $\alpha \in \sigma^{OSSB}$ implies $\alpha \in V(v_T, \lhd)$ and, for node v_{T-1} at the last stage of the game we can induce $\sigma^{OSSB}(v_{T-1}) \subseteq \Sigma V(v_{T-1}, \lhd)$. Then, it must also hold for any predecessor node. This concludes the induction.

2) Example 2 a) in the text shows that the reverse is not true.

Proof of proposition 8 We establish the proposition by proving four claims for some arbitrary node a where we assume that for all successor nodes c of a, $\sigma^{CSSB}(c) = \Sigma V(X(c), \lhd)$ has been established.

Claim 1. Assume that for all $v \in Z$, $\lhd$ establishes a complete order on $X(v)$. Let $\gamma, \alpha \in \sigma^{CSSB}(a)$. Then there are stable sets V and V' with $\gamma \in V(X(a), \lhd)$ and $\alpha \in V'(X(a), \lhd)$.

Proof. α, $\gamma \in \sigma^{CSSB}(a)$ implies that at a, S_γ and S_α, have preferences $\gamma \not\succ_{\vec{S}_\gamma} \alpha$ and $\alpha \not\succ_{\vec{S}_\gamma} \gamma$ where $\gamma \succ_{\vec{S}_\gamma} \alpha$ implies that the coalition which can replace α with γ conservatively prefers α to γ.

By completeness, we cannot have $\gamma \ntriangleleft \alpha$ and $\alpha \ntriangleleft \gamma$. So say that $\alpha \lhd \gamma$. For $\gamma \in \sigma^{CSSB}(a)$, we must have (a) $\gamma \lhd \alpha$ or (b) there exists $c \in \gamma$ and $\phi \in \sigma^{CSSB}(c)$ such that $\gamma \lhd \phi$ and $\phi \lhd \alpha$. In case (a), $\alpha \lhd \gamma$ and $\gamma \lhd \alpha$ imply that for the decisive coalitions in a, $\alpha \not\succ_{\vec{S}_\alpha} \gamma$ and $\gamma \not\succ_{\vec{S}_\gamma} \alpha$ and $\alpha \in \sigma^{CSSB}(a)$. In the case of (b), for the decisive coalition in a, S_γ, $\phi \not\succ_{\vec{S}_\alpha} \phi$ and $\alpha \in \sigma^{CSSB}$.[19]

[19] To see that conditions (a) and (b) are necessary, suppose that neither (a) nor (b) holds. In particular, there is no node $c \in \gamma$ and $\phi \in \sigma^{CSSB}(c)$, such that $\gamma \lhd \phi$ and (in node a) $\phi \lhd \alpha$. If there is no ϕ that blocks γ, then γ is the unique stable continuation following a deviation from α in a and, because $\alpha \lhd \gamma$ and $\gamma \ntriangleleft \alpha$, we must have $\gamma \succ_{\vec{S}_\gamma} \alpha$, contradicting that $\alpha \in \sigma^{CSSB}$. Next, suppose that there is $\phi \in \sigma^{CSSB}(c)$ and $\gamma \lhd \phi$ but $\phi \ntriangleleft \alpha$. In this case, $\phi \succ_{\vec{S}_\phi} \alpha$, and all stable continuations of the deviation in a from α are strictly preferred by S_a to α, contradicting that $\alpha \in \sigma^{CSSB}(a)$.

In case (a), we have stable sets $\gamma \in V(X(a), \triangleleft)$ and $\alpha \in V'(X(a), \triangleleft)$. In case (b), we have $\gamma \in V(X(c), \triangleleft)$ and $\phi \in V'(X(c), \triangleleft)$ corresponding to $\gamma \in V(X(a), \triangleleft)$ and $\alpha \in V'(X(a), \triangleleft)$. $\square$

Claim 2. Assume that for all $v \in Z$, $\triangleleft$ establishes a complete order on $X(v)$ and $\{\rightarrow\}$ is symmetric. If there are stable sets $V(X(a)), \triangleleft = \{\gamma\}$ and $V'(X(a)), \triangleleft = \{\alpha\}$ then $\sigma^{CSSB}(c) = \{\gamma, \alpha\}$.

Proof. By completeness, we cannot have $\gamma \ntriangleleft \alpha$ and $\alpha \ntriangleleft \gamma$. So say that $\alpha \triangleleft \gamma$ and the stable sets can be supported if (a) $\gamma \triangleleft \alpha$ or (b) there exists $c \in \gamma$ and $\phi \in V'(X(c), \triangleleft)$ such that $\gamma \triangleleft \phi$ and $\phi \triangleleft \alpha$.

In case (a), invoking symmetry, $S_a = S_\gamma = S_\alpha$ so that $\alpha \triangleleft \gamma$ and $\gamma \triangleleft \alpha$ if and only if $\gamma \sim_{S_a} \alpha$ and, hence, $\gamma \not\succ_{\vec{S}_\gamma} \alpha$ and $\alpha \not\succ_{\vec{S}_\gamma} \gamma$.

In case (b), invoking symmetry at node a, $\alpha \triangleleft \gamma$ and $\phi \triangleleft \alpha$ implies $\alpha \not\succ_{\vec{S}_a} \gamma$ and $\gamma \not\succ_{\vec{S}_a} \alpha$. $\square$

Claim 3. In the case where $\triangleleft$ is complete but $\{\rightarrow\}$ is non-symmetric, $\Sigma V(X(a), \triangleleft) \nsubseteq \sigma^{CSSB}(a)$.

Proof. Assume $V(X(a), \triangleleft = \{\gamma\})$ and $V'(X(a), \triangleleft) = \{\alpha\}$. In the case of non-symmetry, $\alpha \triangleleft \gamma$ and $\gamma \triangleleft \alpha$ is compatible with $\gamma \succ_{\vec{S}_\gamma} \alpha$ but $\alpha \sim_{\vec{S}_\alpha} \gamma$ so that $\gamma \in \sigma^{CSSB}(a)$ and $\alpha \notin \sigma^{CSSB}(a)$. This proves the claim. $\square$

Claim 4. Assume that $\triangleleft$ is complete and $\{\rightarrow\}$ is symmetric. Then $\alpha \in \sigma^{CSSB}(a)$ implies and is implied by $\alpha \in \Sigma V(X(a), \triangleleft)$.

Proof. $\alpha \in \sigma^{CSSB} \Leftrightarrow$ [there does not exist a deviation to $c \in \gamma$, $c \notin \alpha$, such that for all potential continuations from c, γ, $\phi \in \sigma^{CSSB}(c)$, the coalition effective in the deviation in a, S_a has preferences $\gamma \succ_{\vec{S}_a} \alpha$ and $\phi \succ_{\vec{S}_a} \alpha$].

$\Leftrightarrow$ at least one: [$\gamma \not\succ_{\vec{S}_a} \alpha$] or [$\phi \not\succ_{\vec{S}_a} \alpha$].

$\Leftrightarrow$ One of the following must be true:

(aa) $\alpha \ntriangleleft \gamma$ and $\alpha \ntriangleleft \phi$;

(ab) $\alpha \triangleleft \gamma$ and $\gamma \triangleleft \alpha$ (or the corresponding condition for ϕ);

(ac) $\alpha \triangleleft \gamma$ and there is $\phi \in \sigma^{CSSB}(c)$ such that $\gamma \triangleleft \phi$ and $\phi \triangleleft \alpha$ (or the corresponding condition with γ and ϕ switching places).

$\Leftrightarrow \alpha \in V(X(a), \triangleleft)$. $\square$

As the assumption $\sigma^{CSSB}(c) = \Sigma V(X(c), \triangleleft)$ trivially holds for terminal nodes, each claim follows by induction. Finally, example 6 with preference order (b) in the text demonstrates that $\sigma^{CSSB}(a) \nsubseteq \Sigma V(X(a), \triangleleft)$. This establishes the proposition.

Proof of proposition 9 To show that CSSP neither includes nor is included by OSSB, we provide two examples:

a) Show that CSSP does not include OSSB

$\alpha \in CSSP$ but not in OSSB: Consider example 2 with preference order (a), i.e., $\gamma \succ_{S_a} \alpha \succ_{S_a} \delta$. In this case, the unique OSSB is $\sigma^{OSSB} = \{\gamma\}$ but there are two CSSP: $V^1(\triangleleft^Y) = \{\alpha, \underline{\delta}\}$ and $V^2(\triangleleft^Y) = \{\gamma\}$.

b) Show that OSSB does not include CSSP

Consider an extension of figure 2 where a has a direct predecessor a_0. S_{a_0} may decide to move to node a or it may choose π. The payoffs for S_{a_0}, S_a, S_c are $u(v^\alpha) = (4, 2, 1)$, $u(v^\delta) = (3, 1, 1)$, $u(v^\gamma) = (3, 3, 1)$ and $u(v^\pi) = (3, 1, 1)$.

$\pi \in OSSB\,(a_0)$ but not in $\Sigma V\left(a_0, \triangleleft^Y\right)$: $\alpha \in V\left(a_0, \triangleleft^Y\right)$ because all continuations are weakly preferred to π but α is not dominated by π via $\triangleleft^Y$, hence $\pi \notin V\left(a_0, \triangleleft^Y\right)$. But with α blocked in OSSB, $\{\pi, \gamma\} = \sigma^{OSSB}\,(v_a)$.

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