Coalition Formation Processes in Real Time: Generalizing Subgame Perfection for Dynamic Paths

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Preliminary Version

Abstract

In this paper we develop the stable set defined for dynamic paths as a solution concept for coalition formation processes in real time. We introduce a dominance relationship $\lhd$ *which at each node selects weakly preferred paths. In the case where a singleton moves at each node, our solution includes all subgame perfect paths. Refining the concept by focusing on undominated paths (in the conventional sense), each stable set corresponds to a subgame perfect equilibrium of the corresponding game.*

Keywords: *Coalition Formation, Stable Set, Farsightedness, Dynamic equilibrium*

JEL codes: *C71, F15.*

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1 Introduction

How should we think about coalition formation processes in real time? Coalition formation processes in an atemporal environment typically assume that agents are only willing to move along such paths along which they improve their well-being.[1] Coalition formation processes in real time such as introduced by Konishi/Ray (2003) - although focusing on absorbing states - make a similar assumption about admissible paths. In an atemporal environment, where agents may travel any path, this assumption is innocuuous if paired with an assumption about players' assessment of other players' subsequent moves: Players are either "optimistic" and potentially ignore possible threats posed by moves of other players or they are "conservative" and think adverse moves by other players are a distinct possibility. Yet when analysing coalition formation processes in real time, the focus on indirectly dominating paths implies that agents who participate in the process are farsighted but unsophisticated when it comes to assessing threats. Thus, when we are interested in the path leading to an equilibrium state, analogous considerations to those taken into account in the non cooperative concept of subgame perfection may also be invoked in a coalition formation process.

In a parallel development, a number of recent papers have shown how reference to an explicit bargaining protocoll can obtain insights into coalition formation processes: Gomes/Jehiel (2005) have recently shown that depending on the bargaining protocol and in the presence of negative externalities between coalitions inefficient states can be stable in Markov perfect equilibrium. The dynamic path may involve enduring a temporary inefficiency even when agents can anticipate the inefficiency and can write a contract which avoids this outcome. Alimbekov/Madumarov/Pech (2017) demonstrate in a model with transferable utility, that an open-loop bargaining protocol in the sense of Baron/Ferejohn (1989) allows agents to internalize the negative externality. More generally, Acemoglu/Egorov/Sonin (2012) show in a dynamic environment with transfers that, while inefficient processes are possible, there exist bargaining protocols with which the dynamic system immediately moves into the dynamically stable state.

If a solution concept for a coalition formation process is to make predictions for an institution-free bargaining environment it should, therefore, allow agents to negotiate away from inefficient situations to the extent that the inefficiency is the result of a credible threat. Yet while subgame perfection offers a convincing solution for non cooperative games which establishes the credibility of threats and accounts of them, there is no equivalent notion for coalition formation processes.

In our leading example a formateur has to find an optimal sequence in which to

[1] Such paths are said to indirectly dominate their starting point, see Chwe (1996), Xue (1998) or Pech (2012, 2015).

form a customs union in an environment where coalitions exert negative externalities. In this example, players may be jointly better off by immediately accepting the absorbing state if this is proposed to them rather than going through a period of punishment when the formateur carries out his threat to unleash external effects. The consequences of such threats are ignored in standard models of dynamic coalition formation.

It turns out that our solution concept − the stable set defined on dynamic paths − is capable of taking account of such strategic considerations in a similar way as subgame perfection does in extensive form games. Our modelling approach allows agents to entertain myopia or (sophisticated) farsightedness and is flexible enough to incorporate different institutional constraints.

Section 2 introduces three motivating examples. Section 3 compares our modelling approach to the literature. Section 4 sets up our dynamic coalition formation model. Section 5 provides first results. Section 6 applies our results. Section 7 discusses the relationship of our solution concept with subgame perfect equilibrium. Section 8 presents a refinement of our solution concept.

2 Applications

2.1 Customs Union Formation

Consider the following customs union formation game inspired by Aghion, Antras and Helpman (2007) which is played over two periods with a status quo point a_0 as depicted in figure 1. If each agent stays singleton − corresponding to the vertices labelled "a" with time indices 0, 1 and 2 − each realizes a value of 1. The total value of the grand coalition is 6 and the total value if coalition {A,B} forms the customs union is 4. Moreover the core customs union {A,B} has an externality on C who realizes a pay off of zero. Efficient outcomes are in node c_1 or d_1 where the grand coalition forms in the first period. c_1 can be realized without side payment but d_1 involves a sidepayments between the members of the grand coalition. Possible moves by coalitions are illustrated in figure 1. The inducement correspondences - solid arrows - correspond to moves by some coalition. The broken arrows represent the default path: In any node, if no coalition moves, the empty coalition brings about the default path in that node. Pay offs associated with a position (such as u_d in d_t, $t = 1, 2$) are realized each time the position is realized. Agents have a common discount factor $\delta = 1$.

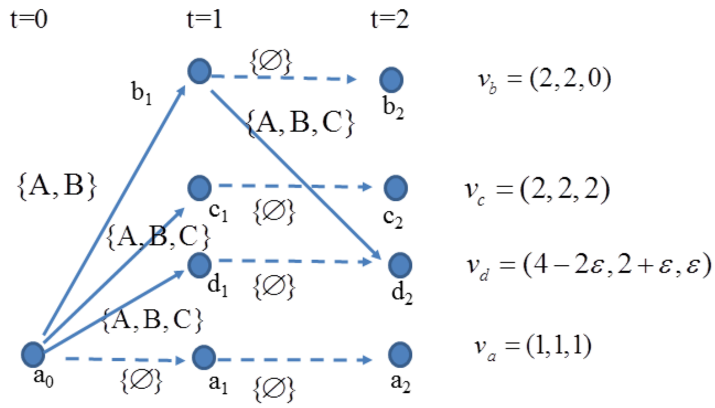

Figure 1: Customs Union Formation Game

A and B prefer the inefficient path $a_0b_1d_2$ to the efficient "equal split" path $a_0c_1c_2$. This is the path which indirectly dominates the status quo point in the sense that the coalition which moves in each point predicts that a subsequent coalition will be willing to move to a node such that the resulting path improves on the default path. The interpretation is that because of the externality, once b_1 is reached, C will be happy to participate in a move to point d_2 in period $t = 2$. This path also corresponds to the solution obtained by Aghion, Antras and Helpman in a framework which assumes a bargaining protocol with A as proposer where proposals are put against the current status quo in $t = 0$ and $t = 1$ and can only be accepted or rejected in a single round of bargaining.[2]

The argument against this solution is that C should realize, that if the path $a_0d_1d_2$ is proposed to her, she is better off than with the path $a_0b_1d_2$. Alimbekov/Madumarov/Pech (2015) argue that if the proposal maker chooses as bargaining protocal an "open-loop" rule and proposes the core customs union as a contingent outcome in the first round of bargaining this will pursuade player C to

[2]This protocol has been labelled "closed-loop" in Alimbekov/Madumarov/Pech (2015).

accept d_1 in the second round of bargaining.[3] It, therefore, makes sense to consider this solution also as a possible outcome in an institution-free bargaining process.

Ultimately, it would seem desirable to have a solution concept for cooperative dynamic games which allows to take into account the same kind of reasoning - here the availability of a credible threat to create a core customs union - which would enter agents' considerations if the game were modelled as an extensive form game with a distinct institutional structure. Whilst different institutions may still support different - and some of them inefficient - solutions, the focus on paths which indirectly dominate the default path rules out the consideration even of credible threats.

2.2 Stability of Free Trade Agreements

Our argument that with no further frictions, the efficient state should immediately obtain, does not negate the result that the efficient state itself might be "unstable" in the sense that coalitions of agents would want to move away from this state. This is a well-known result in a public goods context: If smaller coalitions than the grand coalition are stable and a free rider enjoys positive externalities from such a coalition, the grand coalition will not form (see Ray, 2007). But as Gomes/Jehiel (2005) have shown,[4] the same problem may undermine the stability of the grand coalition in cases where formation of subcoalitions inflict negative externalities on a former partner: In this case, threatening with the externality may be used to lower demands of one member of the grand coalition.

Consider the following variant of our example of section 2.1: Assume that in state a_0, the grand coalition has formed and pay-offs are $(2, 2, 2)$. Any two agents may form a subcoalition with the pay off configuration for the two coalition members and the outsider $(2, 2, 0)$. From our previous argument, if two members threaten with their departure, the third member should immediately accept a lower pay-off. Thus, $\{A, B\}$ could propose the grand coalition and the pay-off vector $(4 - 2\epsilon, 2 + \epsilon, \epsilon)$ against the grand coalition and $(2, 2, 2)$. Similarly, $\{B, C\}$ could propose $(\epsilon, 4 - 2\epsilon, 2 + \epsilon)$ and $\{A, C\}$ could propose $(2 + \epsilon, \epsilon, 4 - 2\epsilon)$. It is

[3] Gomez/Jehiel observe that their inefficiency result only obtains if agents cannot sign general spot contracts where allocations and transfers can be made contingent on players' responses.

[4] Gomes/Jehiel (2005) provide a general inefficiency result which applies to Markov-stable states in an open-horizon model, stating that the set of stable (ergonic) states involves inefficiencies if a move away from some efficient state by at least two agents involves negative externalities on agents who are not effective in brining about this move. Their argument does not directly apply to our examples as they focus on equilibrium proposals of a proposal maker process yet the example may be used to check whether in a particular situation an agent has a strategic move which may induce another agent to accept a lower pay-off. Typically, this will be the case if the agent has a move which inflicts a lower pay-off on an outsider.

evident, that the situation in node a_0 is unstable as different coalitions would want to move away from a_0 and realize different outcomes.

2.3 Political Institutions

Institutional design affects the game in figure 1 in the form of rules about which set of agents can move in any node. Farsightedness of the coalition who can move away from an inefficient state may prevent such a move if the coalition foresees that the coalition who is effective in the efficient state would like to move away from it. The following example is due to Acemoglu/Egorov/Sonin (2012): The status quo (monarchy) gives the elite E and the middle class M a pay-off of 1 each. Both the elite and the middle class prefer constitutional monarchy where each gets a pay-off of 2. Yet in constitutional monarchy, the middle class would opt for full democratization with an associated pay-off for E and M of $(0, 3)$. This game may start in any period and indefinitely stays in the democracy node once it is reached. Clearly, if E is sufficiently patient, it does not want to move away from a because its long-term pay-off from moving to c - taking into account the subsequent movement to b is lower than the pay-off from staying in a.

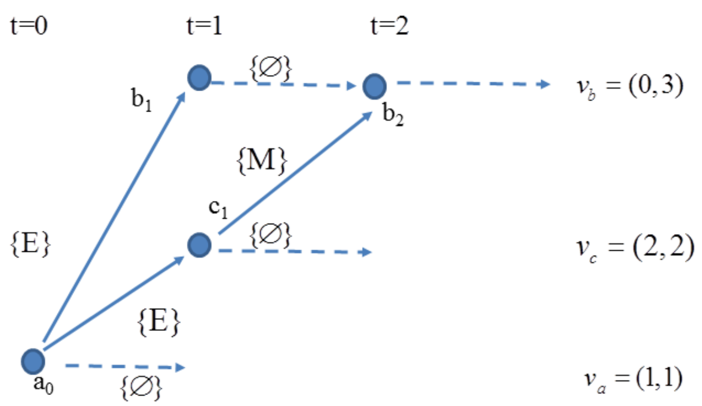

Figure 2: The Democratization Game (Acemoglu/Egorov/Sonin, 2012)

As the examples demonstrate, our concern here is with inefficiencies from ignoring credible threats on a path to a dynamically stable state rather than with strategic moves which keep the economy away from a dynamically stable state.

3 Comparison to the literature

Konishi/Ray (2003) and Gomez/Jehiel (2005) model Markov-perfect strategies in a stationary environment with probabilistic moves and randomized strategies. The present model is more general in that it does allow for non stationary environments and allows for consideration of strategic choices which are not Markov-perfect but it is less general in that it does not allow for mixed strategies.

Greenberg (1989) and Tadelis (1996) analyse optimistic social standards of behavior (OSSB) in tree situations corresponding to simple game trees or repeated games. Greenberg shows that the unique OSSB is a (weak) subset of the set of subgame perfect equilibrium paths (where paths losely correspond to strategies). Tadelis shows that the OSSB for a repeated game is a refinement of subgame perfection with desirable properties such as Pareto-dominance. The framework developed in this paper defines time paths where agents − in each non-trivial node − choose between different continuations of these time paths.

Xue (1998) analyses social standards of behavior in a graph situation. He assumes that in a non trivial node agents can "defect" to a different path. The consequence of this modeling choice in the theory of social situations is that if agents can defect from either of two different paths − a setting where subgame perfection would make a non-unique recommendation − the solution is quiet on which path agents follow, i.e. it assigns the empty set as solution in such a node. The approach followed in this paper allows for the existence of multiple stable sets which more closely corresponds to the idea of subgame perfection.

As we show, our solution generally exists in graphs where at each node only one agent moves and makes a − possibly non-unique − recommendation. A solution fails to exist in the case where the preference relationship on paths is cyclical − for example involving three different coalitions wanting to move from one node along three different paths − as in this case the stable set does not exist.

Finally, Huang and Luo (2008) establish sequentially stable equilibrium which coincides with the largest stable set and yields all subgame perfect paths of play. While Huang and Luo focus on refining subgame perfect equilibrium, our focus is on refining a (slightly) larger set than the set of all sugame perfect paths and in establishing each refined set as selection of paths corresponding to exactly one subgame perfect equilibrium. In blunt terms, we exactly "reinvent" subgame perfect equilibrium as a special case of a solution method for coalitional games.

4 The model

We consider an n-person game G of perfect information without chance moves. Let Z be the set of nodes and Z_t be the finite set of nodes which correspond to period t. X is the set of feasible paths connecting these nodes. Let $x(v_t)$ be the set of paths originating in node v_t and $X(v_t)$ the set of paths which can be reached from node v_t, including paths in $x(v_t)$. Agents have preferences $\succsim_i (v_t)$ defined on $X(v_t)$.[5] This includes the case of hyperbolic discounting of pay-offs accruing in the nodes which are reached along paths starting in v_t as a special case. Feasibility of paths may reflect institutional or factual relationships between situations over time. For convenience we assume that a path assigns at least one vertex to each time period subsequent to the path's origination. Let $a \in Z_t$ and b denote two nodes. An effectivity relation $a \xrightarrow{S} b$ signifies that in situation a, coalition S can bring about situation b. $a \xrightarrow{S} b$ implies that $b \in \Omega(a) \subset Z_{t+1}$, the set of successor nodes of a. Moreover, the arc ab is a subsection of some path.[6] S may be empty in which case b is a vertex on the default path.

Definition 1. A path ${}_a\gamma$ starts in decision node a and consists of a sequence of successive nodes where each node has one predecessor and one successor except for terminal nodes. Successive nodes are connected by an effectivity relation $\xrightarrow{S}$. Paths diverge at every non trivial decision node and never merge.

Note that because all paths orginate in a decision node, no path exclusively consists of an end node. Each possible move of a coalition must belong to some path and nodes which can be induced from one another must be successive in time. Moreover, if no coalition moves in a node, a default path is followed as defined in definition 3.

Definition 2. A path ${}_a\gamma \in X(a)$ dominates path $\alpha \in X$ via $\triangleleft$, i.e. $\alpha \triangleleft {}_a\gamma$, if there is $c \in {}_a\alpha$, $c \notin \alpha$, and S such that $a \xrightarrow{S} c$, and for the truncated paths ${}_a\alpha$ and ${}_a\gamma$: ${}_a\gamma \succsim^S {}_a\alpha$.

So ${}_a\gamma$ dominates α if there is a junction a at which path ${}_a\gamma$ diverges from α and there are agents which are capable of and willing to veer off the path. We do not impose any restriction on the coalition T which is effective in $a \xrightarrow{T} b$ and which may be the empty coalition. In a game tree application, typically T coincides with S. Note that although X contains ${}_a\alpha$ and α as elements, ${}_a\alpha$ does not dominate via $\triangleleft$ the entire path α.

[5] $x(v_t)$ is the set of actions available at node v_t while $X(v_t)$ may be interpreted as available paths - corresponding to strategies - constituting a subgame starting in node v_t.

[6] With a slight adjustment of definition 1 we can also consider situations where "procedural" moves to a node b in the same time-indexed set of situations Z_t as node a are possible.

At any particular node, the domination relation may be cyclical. Cyclicality poses a problem for existence except in the case where there is one path originating in a which dominates all others, so we adjust our definition of cyclicality of $\triangleleft$ accordingly:[7]

Definition 3. $\triangleleft$ is cyclical at node a if there are $K > 2$ paths $\alpha_k \in x(a)$ such that (1) there is a sequence $\alpha_k \triangleleft \alpha_{k+1}$, $k = 1, ..., K-1$, with $\alpha_K \triangleleft \alpha_1$ and (2) there is no path α_j such that $\alpha_j \succsim^{S_j} \alpha_k$ for all $k = 1, ..., K$.

In many applications it is natural to assume that a default path exists which is followed in v_t in the case where no coalition moves at v_t. Formally, in this case we assume that the empty set of agents is effective for the default path ω.[8]

Definition 4. In each node a there is one path which originates and for which the empty coalition is effective. We call this path default path and use $\omega(a)$ as a label for this path.

We assume that the default path dominates another path α in some vertice if one agent necessary to bring about α prefers the default path:

Definition 5. Let ${}_a\alpha$ be a path originating in node a with effective coalition S and ${}_a\omega$ be the corresponding default path. If ${}_a\alpha \not\succ^S {}_a\omega$ then ${}_a\alpha \triangleleft_a \omega$.

In the appendix we consider an alternative definition where the default path dominates a path at some vertex if it dominates all paths originating in this vertex. We show that replacing definition 4 with the alternative definition does not affect the solution set when there is no cycle.

The notion of a default path is natural in a dynamic environment. However, for our analysis we can treat the default path as any other path and we could even suspend with the default path altogether without affecting any of the results of the paper.

[7] With a cycle of paths, stable set is not unique or does not exist. In the case of an uneven number of paths in the cycle, stable set does not exist. With an even number of paths exceeding 2, it may assign more than one path to each stable set. In the latter case we may interpret the inclusion of different paths as representing contemplated possibilities rather than actual actions (I am grateful to Anne van den Nouweland for this suggestion). Another possibility is to say that either there is a mixed strategy with pay-offs $\Sigma_{k=1}^{K/2} s_{2k-1} U(\alpha_k) > \Sigma_{k=1}^{K/2} s_{2k} U(\alpha_{2k})$, or vice versa, or both mixed strategies are admissible (where the s_k's are chosen to maximize the product of utiltity increments over the alternative).

[8] The default path takes the place of the default position in stationary models of coalition formation where agents want to move away from a position if all agents necessary to effect the move approve of it. The default path allows us to capture the case of repeated games of Tadelis (1996) where after each termination node of the stage game another, equivalent, stage game starts and the stationary case of Konishi/Ray (2003) and Gomez/Jehiel (2005) where agents reach a strategically equivalent position in the next period if they don't move away from a node.

Like for any other path, we have to exclude the possibility of the default path being included in a cycle:

Example. 1. In node a, $\{1,2\}$ can induce α with pay off vector $(2/3, 1/3, 0)$, $\{2,3\}$ can induce β with pay off vector $(0, 2/3, 1/3)$ and the default path is associated with the pay off vector $(1/3, 0, 2/3)$.

Here, $\alpha \lhd \beta$, $\beta \lhd \omega$ and $\omega \lhd \alpha$. Because 3 prefers the default path to β, $\beta \not\succ^{\{2,3\}} \omega$, hence $\beta \lhd \omega$. ω obtains if neither $\{1,2\}$ nor $\{2,3\}$ forms to bring about α or β.[9] 1 and 2 would want to replace ω with α, but α itself is vulnerable to β while β may fail because of 3's objection - resulting in a cycle.

Next, consider the case where no coalition which is effective for a path starting in some node, actually wants to move along the path for which it is effective. In this case, the default path may dominate the other paths via $\lhd$ but this is neither guaranteed nor is it crucial for our results:

Example. 2. In node a, S can induce ${}_a\alpha$ and T can induce ${}_a\beta$. The preferences are ${}_a\beta \succ^S {}_a\alpha$ and ${}_a\alpha \succ^T {}_a\beta$.

In this example, no coalition wants to travel along the path for which it is effective. We neither have $\beta \lhd \alpha$ nor vice versa, hence α and β are in the set of paths which are undominated via $\lhd$.

Definition 6. Let $(X, \lhd)$ be an abstract system. $V \subset X$ is stable if it is internally stable and externally stable. It is internally stable if $\alpha \in V$ implies that there is no $\beta \in V$ such that $\alpha \lhd \beta$. And it is externally stable if for all $\gamma \in X \backslash V$ there is $\alpha \in V$ such that $\gamma \lhd \alpha$.

5 Results

We present our existence result for the case where at each node possibly different coalitions move. The case of where only single agents move at each node single agents move is a special case where preferences on paths are transitive.

Lemma 1. *Assume $\forall v^i_{t+1} \in \Omega(v_t) : \exists_{t+1}\alpha_i \in V(X(v^i_{t+1}))$. Then $V(X(v_t))$ exists if one of the following conditions is fulfilled: (a) the relation on α_i is transitive. (b) At all nodes the cardinality c of paths included in a cycle $\alpha^i \lhd \alpha^{i+1} mod(c)$ is even or (c) a cycle of paths $\alpha^i \lhd \alpha^{i+1} mod(c)$ of odd cardinality c has at most $c-1$ relationships of the strict form $\lneqq$. Moreover, there is a default path or each coalition has an effective choice (i.e. it can choose one path rather another, such that $\beta \not\lhd \alpha$ implies $\alpha \trianglelefteq \beta$).*

[9]Compare the effectivity relation to the textbook variant of this example where $\{1,3\}$ can bring about ω. In this case, 1 and 2 can and would want to replace ω with α.

1) $\alpha \not\trianglelefteq \beta$ (i.e. β cannot be supported) implies

a) $\beta \precneqq \alpha$ (reflexivity)

b) $\exists \gamma$ such that $\beta \triangleleft \gamma$ and $\alpha \not\triangleleft \gamma$, $\gamma \not\triangleleft \alpha$ (α and γ are supported)

c) $\exists \gamma$ such that $\beta \triangleleft \gamma$ and $\alpha \triangleleft \gamma$ and $\gamma \not\triangleleft \alpha$ (only γ is supported which may be the default path).

2) $\beta \triangleleft \alpha$ and $\alpha \triangleleft \gamma$ implies

(a) $\beta \triangleleft \gamma$ (transitivity, γ is supported)

(b) $\alpha \triangleleft \beta$, $\beta \not\triangleleft \gamma$ (indifference, β is supported)

Proposition 1. *Assume that one of the conditions of lemma 1 holds. Then a stable set for $(X, \triangleleft)$ exists.*

Proof. At any non trivial node v_t (i.e. with $\sharp\Omega \geq 2$) there must be at least two unblocked paths α and β such that for α (and similarly for β) the condition holds $\nexists v_s \in \alpha$, $\gamma \in X(v_s)$ and S such that $v_s \xrightarrow{S} c$, $c \in \gamma$, $c \notin \alpha$ and ${}_{v_s}\gamma \succ^S_{v_s} \alpha$.

Suppose such a path exists. Replace ${}_t\alpha$ with ${}_t\gamma$. Because paths never merge and cycles are ruled out, in each v_t there is at least one path α such that α is not strictly dominated via $\precneqq$.

Hence, the coalition which at v_{t+1} is effective in brining about α may go it and ${}_{t+1}\gamma \triangleleft_{t+1} \alpha$ for any alternative path. By lemma 1, we can collect paths such that $V(X(v^\alpha_{t+1}))$ is externally and internally stable.

Finally, at v_t, either S^α weakly prefers α to β and the default path or S^β weakly prefers β to α and the default path.

Note: If neither prefers the own path and there is no default path, then there is no domination relation at v_t, hence all paths are collected.

□

Definition 1 establishing the dominance relation ensures that if a path is dominated by another path agents could travel this other path. If they are unwilling to do so, this other path is dominated. So the proposition holds for paths of infinite length $-$ as long as the length is countable and we can determine a continuation path at each point at which a dominating path branches off.

Note that there are no other conditions which we need to impose such as boundedness of pay offs: Assume that the path α and ω are both associated with a pay off of ∞ for all players. If we assume that agents are indifferent between all paths which yield a pay-off of ∞, each path weakly dominates the other and each path is selected into one stable set. Note that in this case existence is ensured but all paths which indefinitely yield non zero profits may be selected.[10]

[10]Unless another criterion for domination between paths is selected.

6 Application to Our Motivating Examples

In our customs union formation example, path $\{a_0d_1d_2\}$ with a total pay off or $(8-4\epsilon, 4+2\epsilon, 2\epsilon)$ and $\{a_0c_1c_2\}$ with a total pay-off of $(4,4,4)$ do not dominate each other via $\triangleleft$. $\{a_0d_1d_2\}$ dominates $\{a_0^b, b_1, b_2\}$ and $\{a_0c_1c_2\}$ dominates the default path via $\triangleleft$. Hence, there is a unique stable set $V = \{\{a_0d_1d_2\}, \{a_0c_1c_2\}\}$.

Now consider the case where agents are farsighted in the sense used in the coalition formation literature and only paths are considered which indirectly dominate the status quo path (see, e.g., Chwe, 1996, or Xue, 1998). Here the set of admissible paths is reduced[11] to $\widehat{X} = \{\{a_0b_1d_2\}, \{a_0c_1c_2\}, \{a_0b_1b_2\}, \{a_0a_1a_2\}\}$. Because $\{a_0b_1d_2\}$ and $\{a_0c_1c_2\}$ do not dominate each other, we have $V = \{\{a_0b_1d_2\}, \{a_0c_1c_2\}\}$.

7 Relationship with SGPE

We can interpret paths in our game as sequences of actions - represented by edges - in an extensive form game.

Proposition 2. *Assume that all paths are finite, only one agent moves in each node and all nodes other than the terminal nodes give a pay off of zero. The sets of stable sets on X includes the set of subgame perfect strategies of the corresponding extensive form game.*

Proof. Let v_t be a decision node with successors v_{t+1}^k and denominate a_s a succession of vertices on path α. Assume ${}_{t+1}\gamma^k \in SGPE_{t+1}^k \Rightarrow {}_{t+1}\gamma^k \in V(X(v_{t+1}^k), \triangleleft)$ for all ${}_t\gamma^k \in x(v_t)$.

At predecessor v_t, $\alpha \in SCPE_t \Leftrightarrow {}_{t+1}\alpha \in V(X(a_{t+1}), \triangleleft)$ and ${}_t\alpha \succsim^i {}_t\gamma^k$ for all ${}_t\gamma^k \in x(v_t) \backslash {}_t\alpha$. Assign $V_t(X(v_t), \triangleleft) = {}_t\alpha \cup V(X(a_{t+1}), \triangleleft)$. V_t is internally stable because a singleton is assigned at $x(v_t)$ and ${}_{t+1}\alpha$ is stable and V_t is externally stable because ${}_t\alpha$ dominates via $\triangleleft$ all other paths in $x(v_t)$, hence it is stable.

For a predecessor node of terminal nodes, a_{T-1}: $\alpha \in SCPE_{T-1} \Leftrightarrow a_T \succsim^i z_T^k$ for all${}_{T-1}\gamma^k \in x(a_{T-1}) \backslash \alpha \Leftrightarrow \alpha_{T-1}\alpha \succsim^i {}_{T-1}\gamma^k$ for all ${}_{T-1}\gamma^k \in x(a_{T-1}) \backslash {}_{T-1}\alpha \Rightarrow {}_{T-1}\alpha \in V(X(a_{T-1}), \triangleleft)$ which completes the induction.

□

For strict preferences on outcomes, our result coincides with the unique backwards induction equilibrium of the theorem of Kuhn and Zermelo. If agents are

[11] The method of constraining the set of admissible paths to indirectly dominating paths starting in a_0 and applying the definition of stability to this set of paths only gives meaningful results if there is no ambiguity of how coalitions move at subsequent stages.

indifferent between outcomes, mulitple SGPE's obtain and the stable set is not unique.

To illustrate, consider the game tree in figure 3. Path γ connects v_0, v_1^1, v_2^1, path α connects v_0,v_1^2 and v_2^3 and β connects v_0,v_1^1 and v_2^2 and branches off from path α in v_1^2. We denominate the branch $\underline{\beta}$.

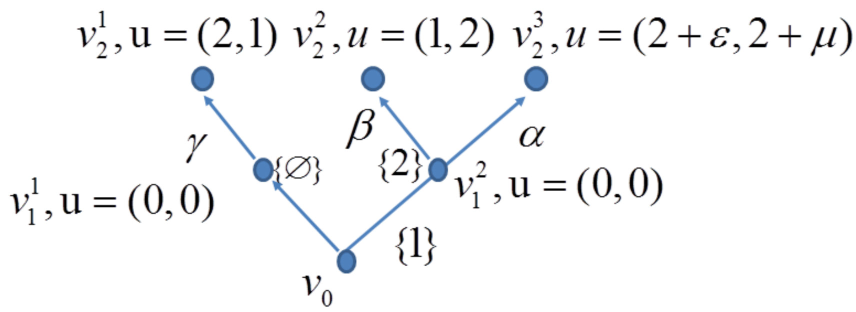

Figure 3: Extensive Form Game

For $\epsilon > 0$, $\mu > 0$ there is one subgame perfect strategy (α). Because α dominates everything else via $\triangleleft$ and is not dominated, $\alpha \in V$ is the only stable path.

For $\epsilon > 0$, $\mu = 0$, there are two subgame perfect strategies: The first consists of γ and $\underline{\beta}$ and the second of α.

β dominates α via $\triangleleft$ and the truncation of α starting in v_1^2, $\underline{\alpha}$, dominates β. Because α dominates γ and $\underline{\alpha}$ also dominates β we have a stable set: $V_1 = \{\alpha\}$. Because $\underline{\beta}$ dominates α, we can select it into V and obtain a second stable set $V_2 = \{\gamma, \underline{\beta}\}$. Note that by our definition, a path originates in a decision node and, therefore, node v_2^2 on its own does not represent a path. Hence, it is not an element of any stable set.[12]

[12]The combination of paths $\{\gamma, \underline{\beta}\}$ is sequentially rational for the fixed belief of player 1 that

For $\epsilon = 0$, $\mu = 0$, γ weakly dominates α in the conventional sense in the normal form of the game and, hence, only γ is selected as an SGPE. Following definition 1, α continues to dominate γ. Hence we still have two stable sets $V_1 = \{\alpha\}$ and $V_2 = \{\underline{\beta}, \gamma\}$.

So the set of stable sets includes paths which correspond to strategies which are not subgame perfect because they are dominated in the conventional sense.

8 Refining the Stable Set

In this section we define a refinement of the stable set, $\widehat{V}(X, \triangleleft)$, which only takes into consideration undominated paths. At the ultimate stage, preceding terminal nodes we define for v_{T-1}: $\{\widehat{V}(X(v_{T-1}, \triangleleft)\} = \{V(X(v_{T-1}), \triangleleft)\}$.

Let $\Omega(v_t)$ be the set of successors of v_t. We can recursively define the refined stable set at v_t as:

$$\{\widehat{V}(X(v_t), \triangleleft)\} = \{V(X(v_t), \triangleleft) \cap [\cup_{v_s^k \in X(v_t)} U(v_s^k)].$$

Note that this definition collects paths which may be followed for each decision node in the subgame $X(v_t)$. Also note that although at no node only the empty set is assigned as solution, by this definition, $\widehat{V}$ assigns the empty set to replace a path which is in V but not in U.[13]

In order to verify whether path which originates in v_t is dominated, we define for each successor node of v_t, $v_{t+1} \in \Omega(v_t)$, the set of (undominated) paths that maybe followed at that stage in some refined stable set:

$W(v_{t+1)}) = \{_{t+1}\gamma |_{t+1}\gamma \in \widehat{V}^j(X(v_{t+1}), \triangleleft)\}$

Definition 7. A considered path $_t\alpha$ is dominated at v_t if $v_t \overset{S}{\to} v_{t+1}$, $v_{t+1} \in \alpha$, $v'_{t+1} \notin \alpha$ and $\exists \gamma \in W(v'_{t+1})$, $v_t \overset{S}{\to} v'_{t+1}$, such that $_{t+1}\gamma \succ^S {}_{t+1}\alpha$ and $\forall_{t+1}\delta \in W(v'_{t+1}), {}_{t+1}\beta \in W(v_{t+1}) : {}_{t+1}\delta \succsim^S {}_{t+1}\beta$.

A considered path is undominated at v_t if it is not dominated. Let $U(v_t)$ be the set of undominated paths originating in v_t.

Definition 8. A path γ dominates α at v_t, $\alpha \blacktriangleleft \gamma$, if $\alpha \triangleleft \gamma$ and $\gamma \in U(v_t)$.[14]

---

player 2 chooses α. Hence, it is part of a sequentially stable equilibrium in the sense of Huang/Luo (2008).

[13]In the example, $_{v_1}\gamma \in \widehat{V}_2(X(v_1))$ but there are two sets $\widehat{V}$ at v_0, $\widehat{V}_1(X(v_0)) = \{_{v_0}\alpha\}$ and $\widehat{V}_2(X(v_0)) = \emptyset$.

[14]Assume that node $v_{t+1} \in \alpha$ is a direct successor of v_t, i.e. $v_{t+1} \in \Omega(v_t)$. A path $_{v_t}\gamma \in X(v_t)$ dominates path $\alpha \in X$ via $\blacktriangleleft$, i.e. $\alpha \blacktriangleleft {}_{v_t}\gamma$, if $_{v_t}\gamma \in U(v_t)$, and there is $v'_{t+1} \in {}_{v_t}\alpha$, $v'_{t+1} \neq v_t$, and S such that $v_t \overset{S}{\longrightarrow} v'_{t+1}$, and for the truncated paths $_{v_t}\alpha$ and $_{v_t}\gamma$: $_{v_t}\gamma \succsim^S {}_{v_t}\alpha$.

We can show that the stable set based on the $\blacktriangleleft$ domination relation returns $\widehat{V}$ but without the elements containing the empty set. We have to focus on the case where there are no cycles because eliminating some domination relations can potentially resolve or cause non-existence problem in a stable set, although not in the corresponding set $\widehat{V}$.

Proposition 3. *Assume that at no node the relation $\triangleleft$ is cyclical. Then $V(X, \blacktriangleleft) = \widehat{V}(X, \triangleleft) \backslash \{\emptyset\}$*

Proof. Using the external stability condition for $V^{\blacktriangleleft}$:

$\alpha \in X \backslash V^{\blacktriangleleft} \Leftrightarrow \exists v_t \in \alpha$ such that $\gamma \in [U(v_t) \cap V(v_t)]$ and $\alpha \triangleleft \gamma$.

Suppose $\alpha \in X \backslash \widehat{V}$. We distinguish two cases:

1. $\alpha \in X \backslash V$:

$\Leftrightarrow \exists v_t \in \alpha$ and $\gamma \in V(v_t)$ such that $\alpha \triangleleft \gamma$. Either a) $\gamma \in \widehat{V}(v_t)$ or b) $\gamma \notin \widehat{V}(v_t)$

In subcase a) $\Leftrightarrow \exists v_t \in \alpha$ and $\gamma \in [U(v_t) \cap V(v_t)]$ such that $\alpha \triangleleft \gamma$.

In subcase b) $\alpha \in \widehat{V}$, contradicting $\alpha \in X \backslash \widehat{V}$.

2. $\alpha \notin X \backslash V$.

Either a) $\alpha \in U(v_s)$, $\forall v_s \in \alpha$ or b) $\exists v_t \in \alpha$ such that $\alpha \notin U(v_t)$.

In subcase a) $\alpha \in \widehat{V}$, contradicting $\alpha \in X \backslash \widehat{V}$.

In subcase b) $\Leftrightarrow \exists v_t \in \alpha$ and $\gamma \in [U(v_t) \cap V(v_t)]$ such that $\alpha \triangleleft \gamma$.

Collecting arguments, $\alpha \in X \backslash V^{\blacktriangleleft} \Leftrightarrow \alpha \in X \backslash \widehat{V}$.

□

Lemma 2. *There must be at least one path in X which is undominated for S.*

Proof. By construction, a path α is dominated in node v_t for coalition S, $\gamma \succ^S \gamma$ iff all paths diverging from γ are (weakly) preferred to all paths starting from α. Hence, the relationship is transitive (no need to exclude cycles).

□

Lemma 3. *The set of undominated paths in X is non empty.*

Proof. The set of undominated paths in X is the core which always exists. Say U^{S_t} is the set of undominated paths for coalition S_t given X_t. Coalition S_{t+s} may have an undominated path $\beta \in X_{t+s}$. Let $Y_{t+s} \neq \emptyset$ be the set of undominated paths for coalition S_{t+s}. $Y_{t+s} \subset U^{S_t}$, hence there is an undominated path for S_t. □

Using the lemmas, the following proposition extends the result of proposition 1 for the refined stable set:

Corollary. *Assume that at no node the dominance relation $\blacktriangleleft$ is cyclical. Then a stable set for $(X, \blacktriangleleft)$ exists.*

Combining U and V, allows us to identify subgame perfect equilibria:

Consider the example in figure 3 again:

For $\epsilon > 0$, $\mu = 0$, neither does α dominate γ nor does γ dominate α. So we have $\Xi(v_1^2) = \{\underline{\alpha}, \underline{\beta}\}$, $\Xi(v_0) = \{\gamma, \alpha\}$ and, hence, $W = \{\underline{\beta}, \underline{\alpha}, \alpha, \gamma\}$. The refined stable sets are $\widehat{V}_1 = W \cap V_1 = \{\alpha\}$ and $\widehat{V}_2 = W \cap V_2 = \{\underline{\beta}, \gamma\}$.

For $\epsilon = 0$, $\mu = 0$, γ dominates α and, hence, $W = \{\gamma\}$. The intersection of V_1 and W is empty, therefore $\widehat{V} = W \cap V_2 = \{\gamma\}$.

So in both cases the recommended behaviour in any node v (i.e. $\Xi(v)$) is the same as the recommendation given by subgame perfection. Overall, W collects all (possibly truncated) paths which may be played in any subgame perfect equilibrium.

Consider the following example taken from Tadelis (1996):

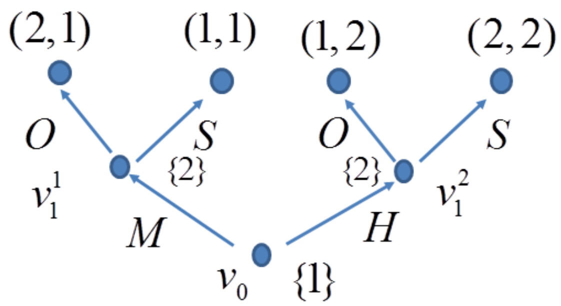

Figure 6: Example

The subgame perfect strategies are $\{S|H, S|M, H\}$, $\{O|M, S|H, M or H\}$, $\{O|H, O|M, M\}$, $\{S|M, O|H, M or H\}$. Note that $S|M$ and $S|H$ together rule out M in a subgame perfect equilibrium in the same way as $O|M$ and $O|H$ together rule out H. Combining the equilibrium strategies to yield entire paths we can write these strategies as $\{SH, S|M\}, \{O|M, SH\}, \{S|H, MO\}, \{O|H, OM\}, \{S|M, OH\}$, $\{O|H, MS\}$.

It is easy to check that $MS \triangleleft HS$ and $HO \triangleleft MO$. This rules out as stable sets $\{MS, S|H\}$ and $\{HO, O|M\}$. Hence, stable sets are $V_1 = \{MO, O|H\}$,$V_2 = \{MO, S|H\}$,$V_3 = \{MS, O|H\}$, $V_4 = \{HS, O|M\}$,$V_5 = \{HS, S|M\}$,$V_6 = \{HO, S|M\}$.

On the other hand, using definition 9 we get $\Xi(v_1^1) = \{O|M, S|M\}$ and $\Xi(v_1^2) = \{O|H, S|H\}$. Thus, neither does the M-path dominate the H-path nor vice versa. So we find $W = \{O|M, S|M, H|M, O|M, MO, MS, HO, HS\}$.

This holds more generally:[15]

Proposition 4. *Consider the stable set for a finite game tree with single player moves $V(X, \blacktriangleleft)$. A path $\gamma \in V(X, \blacktriangleleft)$ if and only if $\gamma \in SPGE$.*

Proof. Let α be a path originating in a and let i be the decision maker (replacing S).

$\alpha \in SGPE \Leftrightarrow [$ There is no deviation by i to some node $c \in \gamma$ such that $\delta \succsim_i \alpha$ for all $\delta \in SGPE] \Leftrightarrow [\gamma \not\succ \alpha] \Leftrightarrow \alpha \in \widehat{V}$.

Assume that the game admits two subgame perfect equilibria, $SGPE_1$ and $SGPE_2$. Let $\alpha \in SGPE_1$ and $\gamma \in SGPE_2$ and let $\cup SGPE$ be the union of $SPGE$'s.

Suppose that $\gamma \succ \alpha$, i.e. there is a deviation by i from $a \in \alpha$ to $c \in \gamma$ and $\gamma \succ_i \alpha$. (Because γ is an $SGPE$ in the game starting in i it must be credible that i plays γ when in a.) Clearly, γ is an $SGPE$ in the remainder of the game, i.e. starting at c and, hence, $\gamma \in \widehat{V}$. But as there is a profitable deviation for i from α, it contradicts that $\alpha \in SGPE_1$.

Hence, $\alpha \in SGPE_1$ implies that $\gamma \not\succ \alpha$ for all $\gamma \in \cup SGPE$ and, in particular, $\cup SGPE = \widehat{V}$.

□

9 Conclusion

In this paper we have defined a dominance relationship $\blacktriangleleft$ which selects weakly preferred paths which are undominated in the conventional sense. The solution concept is generally applicable to coalition formation games, it produces a solution in the absence of decision cycles and each stable set $V(X, \blacktriangleleft)$ corresponds to a subgame perfect equilibrium of the underlying extensive game.

[15] Alternatively, we may define the set W as the $CSSB$ of the corresponding tree situation, see Greenberg (1989), Theorems 8.2.1 and 8.3.1. See also Luo (2006).

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10 Appendix

10.1 The difference between the definition of domination in Xue (1998) and this paper

The notion of domination between paths of definition 1 is different from the standard notion of domination such as in Xue (1998) where a path ξ dominates α if there is a coalition S_γ which is capable of and willing to move from node a on path α to a node on path ξ. Figure 6 illustrates the difference between the notions: The section $\xi = ct(\gamma)$ of path γ is dominated by no other path because no path diverges and the same is true of the section $\psi = bt(\alpha)$ of path α. So assume that S_γ prefers path γ to path α and S_α has the opposite preference. If we use definition 1, path γ dominates α and α also dominates γ. If, on the other hand, we use the standard definition, agents can "defect" from α to undominated ξ and, similarly, agents can "defect" from γ to undominated ψ.

Whilst definition 1 allows for two distinct stable sets (one consisting of α and ξ and the other of γ and ψ), the standard definition only allows for a stable set consisting of the truncated paths ψ and ξ both of which dominate via defections of S_γ and S_α the non truncated paths α and γ and also the "default" or "status quo" point a. In this case, the stable set does not make any recommendations of what agents ought to do in $t = 0$ or, if interpreted within the concept of stable standard of behavior, the solution assigned to node a is the empty set.

The closeness of our solution concept defined for the domination relationship of definition 1 to the concept of subgame perfectness is owed to the fact that at any non trivial node agents "choose" between paths rather than "defect" from paths.

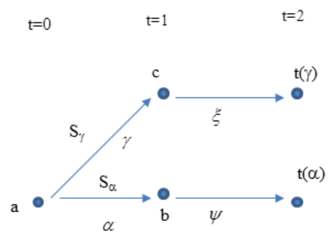

Figure 6: Illustration of the Difference between Definition 1 and the Standard Definition of Domination between Paths

10.2 An alternative notion of domination of the default path

Assume that in node a there are K different paths $\alpha_k \neq \omega(a)$ with effective coalitions S_k. From applying definition 1 and 3, any coalition is effective in bringing about the default path ω which includes at least one agent of the effective coalitions in a, $S_k, k = 1, ..., K$, i.e. the default outcome occurs if no coalition which could move away from a elects to do so. From this, it follows that if $\alpha_k \succ_{S_k} \omega$ fails to simultaneously hold for all $k = 1, ...K$, $\alpha_k \triangleleft \omega$ simultaneously holds for all $k = 1, ..., K$. Focusing on all feasible paths for deciding on domination is unnecessarily restrictive because some feasible paths will never be taken. Therefore, we focus on paths which are included in the solution.

Now consider replacing definition 4 with the following definition 10:

Definition 9. Let α_k be a path originating in node a with effective coalition S_{α_k} and $\omega(a)$ be the corresponding default path. If for all paths α_j which are included in the solution set V, it is true that $\alpha_j \not\succ_{S_{\alpha_j}} \omega$ then $\alpha_k \triangleleft \omega$.

Proposition 5. *Assume that $\triangleleft$ is not cyclical at any node. Replacing definition 4 with definition 10 does not affect the solution set.*

Proof. Assume that apart from the default path ω two separate paths α and β originate in node a with effective coalitions S_α and S_β.

Case 1: $\omega \succsim_{S_\alpha} \alpha$ and $\omega \succsim_{S_\beta} \beta$.

In this case $\alpha \triangleleft \omega$ and $\beta \triangleleft \omega$ and $\omega \in V$ unless ω is blocked by some other path.

Case 2: $\omega \succsim_{S_\alpha} \alpha$ and $\beta \succsim_{S_\beta} \omega$.

Now there is a stable set where $\beta \in V$ unless β is blocked. We say β is blocked if there is some other path γ which diverges from β. So that β is blocked implies that there is some other path $\gamma^* = \underline{\beta} \cap \gamma$.

There are two subcases: $\gamma^* \succsim_{S_\beta} \omega$: In this case, if $\gamma^* \in V$, it replaces β. If $\gamma^* \notin V$, it must be dominated by some other path and we repeat the same step. If there is an infinite sequence of domination (in time), we consider the dominating path η_∞ and it must be that $\eta_\infty \in V$ and $\omega \notin V$.

Applying definition 4 leads to the same result: ω now dominates β but it does not dominate γ^* (or the path in V replacing it).

$\gamma^* \not\succ_{S_\beta} \omega$: In this case, we move to case 1 with γ^* replacing β and $\omega \in V$.

Also here, applying definition 4 leads to the same result because ω will be selected as stable element.

$\square$