Dynamic coalition formation processes: A generalization of subgame perfection based on stable set

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Abstract

We propose two versions of stable set as a solution concept that generalizes subgame perfection for dynamic coalition formation processes. Weakly stable set of paths (WSSP) and weakly conservative stable set of paths (WCSSP) is based on a dominance relationship which derives from a weak preference relation. The new solution concepts are closely related to OSSB and CSSB in the Theory of Social Situations. In game trees with single player moves, WSSP, like CSSB, selects paths that correspond to the set of perfect equilibrium paths. WCSSP, which is based on a more demanding dominance relation, refines this concept and eliminates paths that are dominated in the conventional (strategic) sense. Finally, we generalize WSSP drawing on Delver and Monsuur’s socially stable set, which always exists, and establish weakly socially stable set of paths (WSSSP). WSSSP, like WSSP, generalizes subgame perfection.

Keywords Coalition Formation, Stable Set, Farsightedness, Dynamic equilibrium, Theory of Social Situations

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1 Introduction

How should we think about coalition formation processes over time? When considering a deviation from an existing coalition structure, agents may take into account

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discounted future payoffs and they may transition between states with different constraints in terms of available payoffs or possibilities of forming coalitions (see Konishi and Ray [15]). But agents may also raise the consequences of future play in negotiations. Indeed, what underlies the most fundamental solution concepts in cooperative game theory, which attempts to model free negotiations between agents, is either the absence of threats for the outcomes that enter the solution (as in the core) or the balance between threats and counter-threats (as in stable set). Yet, as Ray and Vohra [26] observed,¹ if one were to incorporate credible threats into a solution concept for games with sequential moves one would have to appeal to some form of subgame perfection and, therefore, one would have to borrow from noncooperative game theory.

In this paper we demonstrate that there is a close relationship between subgame perfection in extensive form noncooperative games and stable set when applied to dynamic paths. We introduce two solution concepts based on stable set: the stable set of paths (WSSP) and the conservative stable set of paths (WCSSP). Both solution concepts are based on a dominance relationship which derives from a weak preference relation, thus departing from the standard definition of stable set: when defining WSSP, we say that a path α dominates another path β if at one node common to both paths, there is a coalition that can instigate α and weakly prefers α to β, hence WSSP uses an optimistic notion of domination. In the case of WCSSP, a path α dominates another path β if the deviating coalition weakly prefers all stable continuations of α to β - a conservative notion of domination. The idea of basing “dominance” on a weak preference relation might seem odd, when we follow the interpretation of dominance in cooperative game theory as “blocking”. A more natural interpretation in our context is to think of a path that dominates as a contemplated possibility and a path that agents may be willing to follow.

When applied to game trees with sequential moves by individual players (which we call simple trees to distinguish them from the general case), the two solution concepts agree with the recommendations of two notions of subgame perfection: the union of (optimistic) WSSP’s coincides with the set of paths that are perfect equilibrium paths (PEP) in the sense of Greenberg (1989).² In simple trees, this solution also coincides with the predictions of Greenberg’s [10] Conservative Stable Standard of Behavior (CSSB). Moreover, there is a one-to-one relationship between perfect equilibrium path and weakly stable set. But like CSSB, WSSP includes paths that are dominated in the conventional (strategic) sense - i.e. where there are alternative paths which the decision maker can choose and which are at least as good

¹Their discussion of example 5.8 of Ray and Vohra [24].
²see also Greenberg, Monderer and Shitovitz’s [11]
as the contemplated path for all rational choices of the other players. Conservative stable set of paths (WCSSP), on the other hand, replicates subgame-perfect Nash equilibrium when applied to simple trees. Our approach helps to further clarify the relationship between subgame perfection and stability (see Luo [18]), and between Optimistic Stable Standard of Behavior (OSSB) and stable set (see Shitovitz [27]): The union of WSSP includes OSSB - which itself refines subgame perfection. If the dominance relation underlying WSSP establishes a complete order at each node - as is implicit in our definition of a simple tree - its predictions coincide with CSSB and, hence, with PEP, although more generally the union of WSSP refines CSSB.

But apart from an exercise in housekeeping, this paper aims at making a substantive contribution to the theory of coalition formation processes over time: weakly stable set of paths and the Theory of Social Situations are both derived from vN-M stable set and, therefore, broadly agree on how to assess a decision situation, but they also suffer from similar deficiencies: in cyclic graphs weakly stable set may not exist while OSSB and CSSB assign the empty set as solution at some nodes. However, stable set has been generalized to deal with situations of non-existence and we demonstrate that such generalization can be applied to WSSP and the resulting solution, like WSSP, generalizes subgame perfection.

Other recent contributions which incorporate farsightedness or dynamics in models of coalition formation are Konishi and Ray [15], who have introduced equilibrium process of coalition formation (EPCF) as a solution concept for dynamic processes, Ray and Vohra [25], who introduce coalition sovereignty into farsighted stable set and Ray and Vohra [26], who address the problem of non-maximizing solutions in farsighted stable set. All these approaches are based on the notion of indirect dominance to exclude outcomes which derives from the Harsanyi [12] critique of myopic stable set: agents are willing to participate in each step along a path if this path ultimately improves their well-being. Formally, for two nodes \( a \) and \( b \), \( b \) indirectly dominates \( a \) or \( b \gg a \), if there exists a sequence of nodes, \( a_0, a_1, ... a_m \) with \( a_0 = a \) and \( a_m = b \), and coalitions \( S_0, S_1, ..., S_{m-1} \) with coalition \( S_i \) capable of moving from \( a_j \) to \( a_{j+1} \) and strictly preferring \( a_m \) over \( a_j \). Invoking indirect dominance makes sense because it ensures that only such threats are raised against the status quo outcome which agents are able and willing to carry out. But focusing exclusively on the indirectly dominating sequence may exclude some of the options from which agents would want to choose in a situation where the indirectly dominating sequence is raised as a credible threat against the status quo.\(^3\) In this case, agents may be

\(^3\)In an atemporal setting this may be considered innocuous because in an open bargaining situation a historical status quo point is generally of less importance. See, however, Pech [20] where coalition formation is conditioned on a political status quo point.
persuaded into a move that preempts the threat even if the move violates the indirect domination criterion. That credible threats are not carried out in equilibrium is such an appealing feature of subgame perfection that we believe it should also be part of a generalization of subgame perfection to dynamic coalition formation processes. The following example demonstrates this problem, for which there is no direct correspondence in standard game trees with sequential moves, in a setting where coalitions with overlapping membership can induce alternative paths.

1.1 Motivating Examples

Consider the following customs union formation game inspired by Aghion, Antras and Helpman [1] which is played over two periods with a status quo point \( a \) as depicted in figure 1 (for \( x = 0 \)).

Example 1. Customs Union Formation The game lasts two periods, each move determines the payoff realization for the current period. In the status quo point \( a \), agents \( A, B, \) and \( C \) each period realize payoff vector \( \{2,2,2\} \). If no coalition moves, agents stay in \( a \). The grand coalition may move along \( \gamma \) to node \( d \) where it realizes each period a payoff vector of \( (4,4,1) \). Or \( A \) and \( B \) may move to \( b \) where the partition \( \{\{A,B\},\{C\}\} \) forms with a payoff vector of \( (3,3,0) \). From node \( b \), the grand coalition may move to \( c \) with a payoff vector of \( (4,4,1) \).

The interpretation is that \( \{A,B\} \), by moving along path \( \alpha \), may form a "core" customs union in node \( b \) which exerts negative externalities on \( C \), attempting to draw \( C \) into the grand coalition with lop-sided payoff distribution.

Total payoffs along the paths are \( \{7,7,1\} \) for path \( \alpha \) and \( \{8,8,2\} \) for path \( \gamma \). The status quo point, over two periods, gives a payoff of \( \{4,4,4\} \). Once node \( b \) is realized, all players gain from continuing on path \( \alpha \) to final outcome \( c \). \( A \) and \( B \) cannot, without \( C \)'s consent, induce path \( \gamma \) which would give both the highest payoff. \( \alpha \) is also inefficient for the grand coalition, when compared to \( \gamma \).

For path \( \alpha \) and path \( \gamma \) there exists a bargaining protocol which implements this path as outcome of a non-cooperative bargaining game: a "closed-loop" bargaining protocol with \( A \) or \( B \) as proposer where proposals are put against the current status quo in the first and, again, in the second period that can only be accepted or rejected in a single round of bargaining implements path \( \alpha \). Under an "open-loop" rule \( A \) proposes the core customs union as a contingent outcome in the first round of bargaining. After \( B \) accepts, in the second round \( A \) asks the other players to agree

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\(^4\)This corresponds to the solution proposed by Aghion, Antras and Helpman (AAH) in an extensive form game of characteristic function bargaining.
on an amendment which offers path $\gamma$. $C$, faced with $\alpha$ as the alternative, has to concede $\gamma$.$^5$

Most cooperative solution concepts agree that $c$ should be realized if the status quo point is $a$: letting $Z$ denote the set of nodes, the farsighted stable set $(Z,\gg)$ includes the end points $c$ and $d$ but only $c$ indirectly dominates $a$ along path $\alpha$. In largest consistent set (Chwe, [5]), a set is consistent if all deviations are deterred. A deviation is deterred if the devitation itself or an indirectly dominating sequence from the deviation results in an outcome in the consistent set which the original deviator does not strictly prefer. The largest consistent set comprises $c$ and $d$ but in point $a$ only the deviation along $\alpha$ to $c$ is predicted. Konishi and Ray [15] introduce equilibrium process of coalition formation (EPCF) where players discount utilities over dynamic paths and engage in a deviation from a status quo state if this maximizes their payoff over the dynamic path. In the example, the only coalition which profits from a deviation from the status quo state $a$ is $\{A, B\}$ along path $\alpha$.

All solution concepts insist that $C$ will never agree to move to $d$ along path $\gamma$, no matter how much $\{A, B\}$ insist that they could bring about the inferior node $b$ until

$^5$See Alimbekov, Madumarov and Pech [2]. Gomes and Jehiel [9] provide a general inefficiency result with a bargaining protocol which is similar to the "closed-loop" variant of the customs union formation example in that it does not allow for contingent agreements.
they actually have done so. The position defended here is that no solution concept should prejudge the outcome of bargaining over the three paths under consideration which, apart from $\alpha$ and $\gamma$, include the default path of staying in node $a$, denoted $\overline{\alpha}$. While $C$ seems to be in a somewhat weaker bargaining position as there is no objection that $C$ can raise against $\alpha$, she is decisive in bringing about the Pareto-better path $\gamma$ and, by withholding consent, can enforce $\overline{\alpha}$ against $\gamma$.

We follow Xue [29] in treating paths as elements of the decision problem and define dominance relations by saying that a path $\alpha$ dominates another path $\beta$, or $\beta \prec \alpha$, if the coalition that can replace $\beta$ with $\alpha$ weakly prefers the outcome that can be reached on $\alpha$ over the outcome that can be reached on $\beta$. Based on this weak dominance relation, we define the (weakly) stable set of paths in node $a$ for all distinct paths starting in $a$ and with stable continuations in the successive node:

$V(a)$ is internally stable, i.e. $\alpha, \beta \in V(a)$ implies that neither $\alpha \prec \beta$ nor $\beta \prec \alpha$, and $V(a)$ is externally stable, i.e. for all $\beta \not\in V(a)$ there is $\alpha \in V(a)$ such that $\beta \prec \alpha$. We get the following dominance relations for our example in node $a$: $\overline{\alpha} \prec \alpha$, $\alpha \prec \gamma$ and $\gamma \prec \overline{\alpha}$. The last relationship obtains because the default path $\overline{\alpha}$ is enforced against $\gamma$ by any one player who is decisive in the coalition that can induce $\gamma$. In this case it is player $C$ who strictly prefers the status quo node and can prevent the move. It is immediate that there is a decision cycle in node $a$: there is no way to select a set of paths such that no path in the set dominates any other path in the set and every path excluded from the set is dominated by a path in the set. In this case, no (weakly) stable set of paths exists.

Stable standard of behavior (SB) agrees with the assessment of the bargaining situation in WSSP but rather assigns the empty set to a node where the solution concept does not give a recommendation of which path to select. In the Theory of Social Situations (TOSS) a path is "dominated" if there is a coalition capable of deviating to another path that is included in the SB such that the coalition "prefers" that deviation. At point $a$, \{A, B\} would want to deviate from $a$ by inducing $\alpha$ and \{A, B, C\} would want to deviate from $\alpha$ by inducing $\gamma$. Applying the same definition of domination to the status quo node, we can say that $C$ wants to deviate from $\gamma$ and force the default path $\overline{\alpha}$. Thus, SB, in node $a$, recommends the empty set.

Non existence or emptiness of the solution is an unsatisfactory expression of what this bargaining situation entails, which is that any of the paths under consideration may be selected but we do not have the information to decide which. In the case of stable set, this problem disappears when there is an even number of alternatives in a

\[\text{In this case, } \alpha \text{ has a stable continuation } \alpha_b \text{ in } b, \text{ because } \overline{\beta} \prec \overline{\alpha}.\]

\[\text{See Greenberg [10]. Xue [29] applies TOSS to a path situation - our discussion refers to his modelling approach. Section 2.2 contains a formal definition.}\]
cycle, in which case stable set returns sets of outcomes that may be realized. There are a number of approaches aiming to resolve the problem of non existence of stable set that are based on the transitive closure of the dominance relation (see Peris and Subiza [22]). We can define the transitive closure of \( \prec \) as follows: \( \alpha_1 \prec \prec \alpha_m \) if and only if there exists a sequence such that \( \alpha_1 \prec \alpha_2, \ldots, \alpha_i \prec \alpha_{i+1}, \ldots, \alpha_{m-1} \prec \alpha_m \). Delver and Monsuur [6] define socially stable set by restricting the transitive closure relation to elements in the set when defining the internal stability condition. Applying this approach to weakly stable set of paths, we can define a weakly (socially) stable set of paths (WSSSP) in point \( a \): for all distinct \( \alpha, \beta \) starting in \( a \), \( \alpha, \beta \in V(a, \prec\prec) \) if \( \alpha \prec \prec V(a) \beta \) implies \( \beta \prec \prec V(a) \alpha \) and for all \( \beta \not\in V(a, \prec\prec) \) there is \( \alpha \in V(a, \prec\prec) \) such that \( \beta \prec \alpha \). In example 1 we obtain the weakly socially stable set of paths as \( V(a, \prec\prec) = \{ \pi, \alpha, \gamma \} \). While WSSSP ”overgeneralizes” WSSP, that is it may add paths to the solution even when there is no cyclicity, it makes the same prediction as WSSP in the case of simple trees which coincides with the prediction of subgame perfection. In this sense, like WSSP, WSSSP generalizes subgame perfection.

Kimya [14] introduces Equilibrium Coalitional Behavior (ECB), an approach which by construction generalizes subgame perfection but differently assesses the decision situation in example 1. The solution is based on a backward induction procedure to evaluate paths and the equilibrium is defined a set of paths from which there is no strictly profitable coalitional deviation. Coalitional deviations may be blocked by a subcoalition of the coalition that can instigate the deviation: \( \alpha \) is blocked by \( \delta \) but \( \delta \) itself is blocked by \( \pi \), so \( \pi \) is the unique ECB. Dutta and Vohra [7] take a different approach to derive this result: in their definition of strong rational expectations farsighted stable set (SREFS) they require that if a state is non stationary (i.e. if it is dominated), it must be dominated via an optimally profitable path in the sense that no subcoalition of the initiator benefits by joining another coalition in bringing about a different outcome. In our case, suppose that \( \{A, B\} \) initiate \( \alpha \). But \( \{A, B\} \) are both better off by joining \( \{A, B, C\} \) in bringing about \( \delta \). In this case, the maximizing coalition does not form because \( \{C\} \) prefers to stay in \( a \).

The following example illustrates the differences between stable set of paths and CSSB. CSSB is an SB that is based on a conservative concept of domination, that is the deviating coalition ”prefers” to deviate if it strictly prefers all continuations of the path to which it deviates which are included in the solution. In contrast, a path is optimistically ”dominated” if the deviating coalition strictly prefers at least one continuation of the other paths which are included in the solution. Consider the following game depicted in figure 2:

**Example 2.** In the root \( a \), coalition \( S_a \) chooses between path \( \alpha \) leading to terminal
node \( v^\alpha \) and path \( \gamma \), leading to node \( c \). In node \( c \), a single player \( C \) can decide between continuing along path \( \gamma \) to terminal node \( v^\gamma \) and moving along the truncation of path \( \delta, \tilde{\delta} \), to terminal node \( v^\delta \). We assume that \( C \) is indifferent between the two terminal nodes. For coalition \( S_a \) we consider two different preference orders:

(a) \( \gamma \succ_s a \alpha \succ_s a \delta \),
(b) \( \gamma \succ_s a \alpha \sim_s a \delta \).

In case (a), there are two subgame perfect Nash equilibria: \( \{ \alpha, \delta \} \) and \( \{ \gamma \} \). Both are also weakly stable: \( V^1(X, \ll) = \{ \alpha, \delta \} \) where \( X \) is the set of all paths. This result obtains because \( \gamma \ll \delta \) and \( \delta \ll \alpha \) and \( V^2(X, \ll) = \{ \gamma \} \) because \( \delta \ll \gamma \) and \( \alpha \ll \gamma \). CSSB assigns as solutions and \( \sigma^{CSSB}(c) = \{ \gamma, \tilde{\delta} \} \) \( \sigma^{CSSB}(a) = \{ \alpha, \gamma \} \): judging conservatively, neither \( S_a \) nor \( C \) would want to deviate from the proposed path. The union of all WSSP on the one hand and CSSB on the other hand and, hence, their predictions, coincide.

In case (b), subgame perfect Nash equilibrium is unique up to the choice of \( C \): \( S_a \) chooses \( \{ \gamma \} \) and \( C \) chooses \( \gamma \) or \( \tilde{\delta} \). \( \alpha \) is weakly dominated for \( S_a \) in the conventional, strategic sense: for all choices of \( C \), moving along \( \gamma \) offers a payoff that is at least as great as \( \alpha \). The stable set of paths continue to be \( V^1(X, \ll) = \{ \alpha, \delta \} \) and \( V^2(X, \ll) = \{ \gamma \} \) while CSSB returns \( \sigma^{CSSB}(c) = \{ \gamma, \tilde{\delta} \} \), \( \sigma^{CSSB}(a) = \{ \alpha, \gamma \} \). Playing \( \alpha \) can be
rationalized if $S_a$ believe that player $C$ is playing $\delta$ with probability one if node $c$ is reached. Unlike CSSB and WSSP, our second solution concept, WCSSP agrees with subgame perfect Nash equilibrium. In general, we find that if players' (including coalitions') preferences establish at each node a complete order on paths, then the predictions of CSSB and WSSP coincide. To show, when they are different, consider the following variant of example 2:

**Example 3.** $S_a$ has the strict preference order of example 2 (a) but at node $c$ coalition $S_\gamma$ can induce $v_\gamma$ and coalition $S_\delta$ can induce $v_\delta$. Both prefer either terminal node to staying in $c$ but in preference order (a) each prefers the terminal node that they themselves can induce while in preference order (b) each prefers the terminal node that the other coalition can induce:

(a) $\gamma \succ_S \delta$ and $\delta \succ_S \gamma$

(b) $\delta \succ_S \gamma$ and $\gamma \succ_S \delta$

In case (a), $\sigma^{CSSB}(c) = \emptyset$: both coalitions want to deviate from the status quo node and from the path that the other coalition prefers. In $c$, CSSB does now not make any prediction - even though the situation changes little from the case of example 2: the unique CSSB of this game has $\alpha \in \sigma^{CSSB}(a)$.

WSSP non uniquely assigns a value to node $c$ as $\gamma \lhd \delta$ and $\delta \lhd \gamma$: $V^1(c, \lhd) = \{\gamma\}$ and $V^2(c, \lhd) = \{\delta\}$. For the overall game, $V^1(X, \lhd) = \{\gamma\}$ and $V^2(X, \lhd) = \{\alpha, \delta\}$ which corresponds to the prediction of subgame perfection in the closely related case of example 2. WSSP does not make a prediction in the case of a decision cycle over an odd number of paths as in the case where in node $c$ we have three decisive coalitions governing over the paths $\beta$, $\gamma$ and $\delta$ with $\beta \succ S_\beta \gamma$, $\gamma \succ S_\gamma \delta$, $\delta \succ S_\delta \beta$. In this case, weakly socially stable set of paths, $V(\ll)$ assigns the set $\{\beta, \gamma, \delta\}$ to node $c$.

In case (b), $\sigma^{CSSB}(c) = V(c, \searrow) = \{\gamma, \delta\}$: while no coalition wants to stay in $c$, none would want to deviate from the path that the other coalition induces. For stable set of paths, the dominance relationships are $\gamma \nLeftArrow \delta$ and $\delta \nLeftArrow \gamma$. Considering the decision in node $a$, $\alpha \in \sigma^{CSSB}(a)$: it is sufficient that $\alpha$ is preferred by $S_a$ to one of the two continuations in node $c$, i.e. $\delta$.

Formally, $\alpha$ dominates $\delta$ but $\alpha$ itself is dominated by $\gamma$. This argument leads to $\{\gamma, \delta\}$ as the unique stable set. In this case $V(\searrow)$ is multivalued and WSSP as an optimistic solution concept excludes $\alpha$: the behavioral implication is that $S_a$ deviates from $\alpha$ in the expectation that $S_\gamma$ prevails at node $c$. This property carries over to weakly socially stable set $V(\ll)$ when it is multivalued because of a decision cycle - but for slightly different reasons and with different implications: consider again the cycle $\beta \lhd \delta \lhd \gamma \lhd \beta$ where $\{\beta, \gamma, \delta\} \in V(c, \ll)$. If $\delta \lhd \alpha$ then all paths in the cycle
are dominated via $\prec_{\gamma}$. At the same time, $\alpha \prec_{\gamma} \gamma$ so that the reverse domination is "balanced".

Example 3 (a) shows that cyclicity of the dominance relation differently affects WSSP, WSSSP and CSSB. The example serves to demonstrate the general result that if WSSP exists and CSSB assigns a non-empty set to each node, the union of WSSP is included in CSSB. If, in each node, $\prec$ establishes a complete order on the set of emanating paths, as in example 2, the union of WSSP and CSSB coincide. Example 3 (b) shows that the condition of completeness is critical.

WSSP may assign a solution where SB fails to do so while WSSSP always does. This adds to the strength of solution concepts based on stability: Ray and Vohra [24] show that SB is capable of resolving the problem of maximality in farsighted stable set which consists in not enforcing that a deviating coalition picks a maximizing deviation. But SB fails to do so when it assigns the empty set at some node. To see this, consider the following example, due to Ray and Vohra [24].

**Example 4.** Figure 3 expands the graph in figure 2 with node $a_0$ inserted ahead of node $a$. In $a_0$, $A$ has to decide whether to move to $a$. In $a$, $\{B, C\}$ decide between $\alpha$ and $\gamma$ but in node $c$, each of them may separately induce their respectively preferred node. Both prefer outcomes that can be reached from $c$ to the outcome from following $\alpha$, $A$’s highest pay off in $v^\alpha$ but she would rather stay in $a_0$ than realizing any of the outcomes that can be reached from $c$.

SB assigns the empty set to node $c$. Thereby, path $\gamma$ ($\delta$) cannot be evaluated and SB assigns path $\alpha$ as solution to node $a_0$. But initiating $\alpha$ by moving to $a$ is irrational for player $A$: while it is unclear which path will be followed at node $c$ where $B$ may have an opportunity to induce $\delta$ or $C$ may induce $\gamma$, both prefer the outcome to $v^\alpha$ and, hence, either $\gamma$ or $\delta$ is going to prevail. But this should, ultimately, deter $A$ from moving away from $a_0$. WSSP resolves this problem: it assigns two WSSP to node $a$: $V^1(a, \prec) = \{\delta\}$ and $V^2(a, \prec) = \{\gamma\}$. Accordingly, $V^1(a_0, \prec) = V^2(a_0, \prec) = \{a_0\}$.

Our paper is organized as follows: section 2 sets up our dynamic coalition formation model and introduces WSSP. Section 2.1 shows existence, section 2.2 establishes the relationship with $SB$, section 2.3 establishes the relationship with subgame perfect equilibrium. Section 3 introduces WCSSP. Section 3.1 shows existence, section 3.2 establishes the relationship with subgame perfection and section 3.3 establishes the relationship between WSSP and WCSSP. Section 4 discusses extensions to unbounded games and cyclic paths and introduces weakly socially stable set as generalization of WSSP. Section 5 concludes.

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Ray and Vohra [26] refine farsighted stable set to address this problem.
2 Weakly Stable Set of Paths (WSSP)

We consider an $n$-person game of perfect information without chance moves. Let $Z$ be the set of nodes. $X$ is the set of feasible paths connecting these nodes and $X(v_t)$ the finite set of paths which can be reached from node $v_t$ including $v_t$ itself. Agents have preferences $\succ_i (v_t)$ defined on $X(v_t)$. Feasibility of paths may reflect institutional or factual relationships between situations over time. Let $a$ and $b$ denote two nodes. An effectivity relation $a \rightarrow^S b$ signifies that in situation $a$, coalition $S$ can bring about situation $b$. $a \rightarrow^S b$ implies that $b \in \Omega(a)$, the set of successor nodes of $a$. Moreover, the arc $(ab)$ is a subsection of some path. Each possible move of a coalition must belong to some path. We denominate $\alpha_{v_t}$ the truncated subsection of $\alpha$ starting in node $v_t$. The option of remaining in node $a$ is designated as path $\overline{a}$.

**Definition 1.** A path $\gamma \in X(a)$, other than the default path $\overline{a}$, starts in node $a$ and consists of a sequence of successive nodes where each node has one predecessor and one successor on that path, except for start and end node of a path. Successive nodes are connected by an effectivity relation $\rightarrow^S$. Paths diverge at every non trivial decision node and never merge.

The assumption that paths never merge is slightly more restrictive than the envi-
vironment in Xue [29]. Essentially, we assume that different histories are represented by different paths as is appropriate for a theory of dynamic coalition formation. The environment considered in this paper is more general than Greenberg’s [10] tree situation - which corresponds to standard extensive form games - in that it allows for different players or coalitions to move in each node and it allows players to receive a payoff not only in the terminal nodes.

**Definition 2.** A path $\gamma \in X(a)$ dominates path $\alpha \in X(a)$ via $\triangleleft$, i.e. $\alpha \triangleleft \gamma$, if there is $c \in \gamma$, $c \notin \alpha$, and $S$ such that $a \overset{S}{\rightarrow} c$, and $\gamma \succcurlyeq_{S} \alpha$.

So $\gamma$ dominates $\alpha$ if at their junction $a$ there are agents which are capable and willing to veer off path $\alpha$. We do not impose any restriction on the coalition $S$ which is effective in $a \overset{S}{\rightarrow} c$ and which may be the empty coalition.

Agents may prefer to remain in the status quo point rather than moving away from it. The default path $\bar{\alpha}$ is realized if no coalition wants to deviate from $a$ and it dominates another path $\alpha$ in $X(a)$ via $\triangleleft$ if $S$ is effective for $\alpha$ and $\alpha \not\succ_{S} \bar{\alpha}$.

By its definition, stable set is free of contradictions: no path in the stable set dominates another path in the set. And stable set accounts of all paths: paths not in the set are dominated by paths in the set.

**Definition 3.** For every node $a \in Z$, the set $V(X(a), \triangleleft) \subseteq X(a)$ is weakly stable if it is weakly internally stable and weakly externally stable. It is weakly internally stable if $\alpha \in V(X(a), \triangleleft)$ implies that there is no $\beta \in V(X(a), \triangleleft)$ such that $\alpha \triangleleft \beta$. And it is weakly externally stable if for all $\gamma \in X(a) \setminus V(X(a), \triangleleft)$ there is $\alpha \in V(X(a), \triangleleft)$ such that $\gamma \triangleleft \alpha$. The weakly stable set of paths, $V(X, \triangleleft)$ assigns $V(X(a), \triangleleft)$ to all all non terminal nodes $a \in Z$ such that $V(X, \triangleleft)$ is weakly internally and weakly externally stable.

Defining the stable set for all "subgames" of $X$ allows us to establish a one-to-one relationship between subgame perfect path and stable set - as both concepts assign a solution to each subgame. Note that $\triangleleft$ does not generically define a partially ordered set because $\alpha \triangleleft \beta$ and $\beta \triangleleft \gamma$ does not imply $\alpha \triangleleft \gamma$. As a consequence, $V(X(a), \triangleleft)$ may be multivalued at some node $a$.

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9Xue [29] admits cyclic graphs and finds that an acyclic graph is a sufficient condition for existence of CSSB and OSSB. Preference or decision cycles, which are also admitted in this paper, result in an empty stable SB in the node where they appear. See also Pech [21] on sufficient conditions for existence of OSSB in cyclic graphs. As discussed in section 4, existence of CSSB in cyclic graphs only extends to WCSSP but not to WSSP.
2.1 Existence

At any particular node, the domination relation may be cyclic.

Definition 4. A graph is cyclic at node $a$ if there are $K > 1$ distinct paths $\alpha_k \in X(a)$ and a sequence that satisfies $\alpha_k \triangleleft \alpha_{k+1}$, $k = 1, \ldots, K - 1$, $\alpha_K \triangleleft \alpha_1$ and there is no relation $\alpha_{k+1} \triangleleft \alpha_k$ for any $k$.

It is well known that Von Neumann-Morgenstern stable set may not exist if a dominance relationship is cyclic and the cycle has odd length. A graph is cyclic according to definition 4 if and only if a graph is cyclic in terms of the strict relationship $\alpha_{k+1} \succ \alpha_k$, $k = 1, \ldots, K - 1$, $\alpha_1 \succ \alpha_K$. Acyclicity is a sufficient condition for existence of a weakly stable set of paths.\(^{10}\) The condition ensures that there is at least one element which is maximal with respect to $\triangleleft$, that is any ”incoming” dominance relation $\alpha \triangleleft \beta$ for $\alpha$ is matched by an ”outgoing” dominance relation $\beta \triangleleft \alpha$. The proof of lemma 1, which established existence of $V(\triangleleft)$ at the last stage of the game, exploits this property:

Lemma 1. For node $v_{T-1}$ in the last stage of the game, assume that $\triangleleft$ is acyclic. Then weakly stable set $V(v_{T-1}, \triangleleft)$ exists.

All proofs are contained in the appendix. The proof of lemma 1 borrows from Gillies [8]. The condition of lemma 1 is in particular fulfilled in the case where in each node a single agent moves and the agents’ preferences establish a complete order on $X(v_{T-1})$.

Definition 2 establishing the dominance relation ensures that if a path is dominated, agents are willing to choose the other path unless this other path itself is dominated. As proposition 1 shows, in the absence of decision cycles at each node, there needs to be at least one path starting in each vertex $v \in Z$ that is undominated in that weakly stable set.

Proposition 1. Assume that at each node $z \in Z$, $\triangleleft$ is acyclic. Then stable set for $(X, \triangleleft)$ exists.

2.2 Relationship between WSSP and Stable SB

Our solution concept is closely related to the Theory of Social Situations (TOSS) as introduced by Greenberg [10] and applied to path situations by Xue [29] and Greenberg, Monderer, Shitovitz [11]. TOSS distinguishes between optimistic stable

\(^{10}\)It is also a sufficient condition for non emptiness of CSSB and OSSB, see Xue, 1998.
standards of behavior (OSSB) and conservative stable standard of behavior (CSSB). A path is an OSSB (CSSB) if there is no deviation from the path such that some (all) possible continuations which are themselves in OSSB (CSSB) are strictly preferred by the deviating coalition (Xue, [29]):

**Definition 5.** \( \sigma \) is an OSSB (CSSB) if for all \( a \in Z \), \( \alpha \in X(a) \setminus \sigma(a) \) \iff there exist \( S \subset N \), \( b \in \alpha \), and \( c \in Z \) such that \( b \xrightarrow{S} c \), \( (\sigma(c) \neq \emptyset) \) and \( \alpha \succ_{S} \beta \) for some (all) \( \beta \in \sigma(c) \).

Because \( \sigma \) includes all paths from which agents do not want to deviate while WSSP can be non unique when it predicts different paths as a solution, the corresponding set for comparison is the union of all WSSP, \( \Sigma V(X, \prec) = \cup_k V^k(X, \prec) \).

**Proposition 2.** Assume that \( \prec \) is acyclic at all \( z \in Z \). Then \( \sigma^{OSSB} \subseteq \Sigma V(\prec) \) but the reverse is generally not true.

In the appendix, we show that for every path in OSSB there is \( V(X, \prec) \) that contains the path. The following counter example shows that the reverse does not generally hold:11

**Example 5.** Consider the game in figure 2 with payoffs for the first player \( A \), who moves in \( a \), and the second player \( C \), who moves in \( c \): \( u(v_a) = (1.5, 2) \), \( u(v_\alpha) = (1, 2) \) and \( u(v_\gamma) = (2, 2) \).

The unique OSSB of this game is \( \sigma^{OSSB} = \gamma \) which is also an WSSP: \( S_a \) expects player \( C \) to choose the favorable path \( \gamma \). There is another WSSP: \( V(\prec = \tilde{\delta}_c, \alpha) \). Hence, \( \sigma^{OSSB} \not\supseteq \Sigma V(X, \prec) \).

The following proposition shows that a path is included in some stable set of paths (WSSP) if it is a conservative stable standard of behavior (CSSB) and that the two concepts coincide in the case where \( \prec \) establishes a complete order on the paths emanating in any node:

**Proposition 3.** Assume that \( \prec \) is acyclic at all \( z \in Z \). If for all nodes \( z \in Z \), \( \prec \) also establishes a complete order on \( X(v) \), then \( \sigma^{CSSB} = \Sigma V(X, \prec) \). If the latter condition is not fulfilled, \( \Sigma V(X, \prec) \subseteq \sigma^{CSSB} \).

This proposition explains why, although OSSB is closely related to stable set (Shitovitz [27]) and, as Luo [18] and this paper show, stability is related to subgame perfection, it is CSSB that coincides with the set of perfect equilibrium paths. While

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11 See also Tadelis [28], who shows that OSSB refines subgame perfection and, hence, CSSB.
stability implicitly assumes optimistic behavior, completeness of the order established by $\prec$ ensures that different paths originating in some node $c$ and included in $\sigma^{CSSB}$ balance each other as threats that can be raised against the other path and, hence, any such path can be excluded from some weakly stable set of paths.

To see the role of completeness, consider again example 3 with preference order (b) where the coalitions moving in node $c$, $S_\delta$ and $S_\gamma$, each prefer the end node that the other coalition can induce and, hence, $\prec$ is not complete: at $c$ we have neither $\delta \prec \gamma$ nor $\gamma \prec \delta$. In this example, $\delta \in \sigma^{CSSB}(c)$ and $\delta \not\succ \alpha$, hence $\alpha \in \sigma^{CSSB}(a)$. Moreover, $\gamma, \delta \in V(X(c), \prec)$, but because $\alpha \prec \gamma$ and $\gamma$ cannot be excluded from any weakly stable set, $\alpha$ is not stable.

In applying stable set, there is the implicit assumption that agents are optimistic in their assessment of continuation of play, hence $\gamma$ is an effective objection to $\alpha$. CSSB, by contrast, assumes that agents are conservative in assessing continuation of play, hence it is sufficient that possible outcome $\delta$ is not preferred to $\alpha$, to deter a deviation from $\alpha$. Now change the example such that a single player $C$ with a complete preference order moves in node $c$. In this case, $\gamma$ and $\delta$ are only in $\sigma^{CSSB}$ if $C$ is indifferent which translates into $\delta \prec \gamma$ and $\gamma \prec \delta$. This is sufficient to ensure that $\alpha$ is included in some weakly stable set.

Because proposition 3 establishes that for a game with single player moves, $\Sigma V(X, \prec)$ coincides with CSSB, it follows from results on CSSB in tree situations that $\Sigma V(X, \prec)$ of a simple tree must coincide with the set of perfect equilibrium paths, $PEP$ (see Greenberg [10]). In the following section we formally show that a path included in $\Sigma V(X, \prec)$ is, indeed, a perfect equilibrium path and that for each subgame perfect equilibrium there is a corresponding weakly stable set of paths.

2.3 Relationship between WSSP and SGE

In order to establish a relationship between WSSP and subgame perfect equilibrium, we interpret strategies in extensive form games as paths, following Greenberg [10] and Greenberg, Monderer and Shitovitz [11] and we focus on game trees with single player moves:

**Definition 6.** Define a simple tree as follows: In each node one agent moves, all nodes except possibly the terminal nodes give a payoff of zero and staying in any non terminal node is not a permissible move.

This definition corresponds to Greenberg’s [10] tree situation. By proposition 3, in the case of a simple tree CSSB and $\Sigma V$ make the same prediction. But as example 2 in the introduction has demonstrated, they may include paths which correspond to
strategies that do not constitute a subgame perfect Nash equilibrium because they are dominated in the conventional (strategic) sense. However, all subgame perfect paths are included in the set of weakly stable set of paths: let \( SGE(v_t) \) the set of all paths originating in \( v_t \) which may be played as part of a subgame perfect equilibrium.

**Proposition 4.** Assume a game can be represented as a simple tree. The sets of weakly stable sets \( \Sigma V(X(v_t)), \triangleleft \) includes the set of subgame perfect paths, \( SGE(v_t) \) of the corresponding extensive form game.

However, as example 2 shows, \( \triangleleft \) makes it "too easy" for a path to dominate and, thus, too many paths are supported in some weakly stable set. In the following section we provide a stronger notion of domination where a path dominates another path if all stable continuations of the former are weakly preferred to the other path. Although it may not be intuitive at first sight that making the domination relation more demanding will result in a refinement of the solution concept, it turns out that with this definition, in simple tree games all paths that are weakly dominated in the conventional (strategic) sense are excluded from any stable set.

To establish the one-to-one relationship between subgame perfect equilibrium and WSSP, we say that a strategy profile of an extensive form game assigns actions \( s_{v_t^k} \) for all non terminal nodes \( v_t^k \in Z \). \( \alpha_{v_t^k} \) is an equilibrium path corresponding to this strategy profile if the sequence of equilibrium actions starting in \( v_t^k \) induce this path. The one-to-one property follows from the proof of proposition 4:

**Corollary.** Assume a game can be represented as a simple tree. For each path in \( \alpha \in SGE(v_t) \) there is one corresponding weakly stable set \( V(X(v_t)), \triangleleft \) with unique solution \( \alpha \in V(X(v_t)), \triangleleft \). Moreover, let \( \{s_{v_t^k}\} \) a subgame perfect strategy profile. Then there exists \( V(X, \triangleleft) \) with \( \alpha_{v_t^k+j} \in V(X, \triangleleft) \) such that \( \alpha_{v_t^k+j} \) corresponds to the sequence of actions assigned to \( v_t^k \).

To see how subgame perfect strategy profile corresponds to a set of paths, consider example 2 (b) with preference order \( \gamma \triangleright_S a \sim_S \delta \). \( \{\gamma\} \) and \( \{\alpha, \delta\} \) are subgame perfect strategy profiles and also weakly stable sets where for the latter \( \delta \) blocks \( \gamma \).

### 2.4 Subgame rationalizability

Because Luo [18] establishes a relationship between generalized stable set and subgame rationalizability as introduced by Bernheim [3] and Pearce [19] we, too, address the relationship between WSSP and subgame rationalizability.\(^{12}\) A strategy profile...
in an extensive form game is rationalizable if there exist conjectures for the individual players about what other players are expected to do at each node, which are consistent with rationality, such that it is optimal to play this strategy profile. Each subgame perfect strategy profile is also rationalizable. If an extensive form game has a unique subgame perfect equilibrium, it is also the unique subgame rationalizable strategy profile but in the case of non uniqueness the set of included strategies diverge: consider part a) of example 2 where monolithic player $S_a$ has the preference order $\gamma \succ_s \alpha \succ_s \delta$, $\alpha$ dominates $\delta$ via $\triangleleft^Y$. The subgame perfect paths are $\{\alpha, \delta\}$ and $\{\gamma\}$. $\{\alpha, \gamma\}$ is subgame rationalizable but neither subgame perfect nor a WSSP: $S_a$ is predicted to move along $\alpha$ but once $c$ is reached, $C$’s actions are not constrained by this conjecture. By contrast, in the case where each separate WSSP makes a unique prediction of play such as in the simple tree environment, WSSP, like Nash equilibrium, postulates that this course of play is self-fulfilling.\textsuperscript{13}

3 Weakly Conservative Stable Set of Paths (WCSSSP)

In this section we define a "conservative" weakly stable set of paths (WCSSP) where agents only deviate from a path $\delta$ to a node $c \notin \delta$, if they prefer to $\delta$ all elements in $Y \subseteq X(c)$, the set of all stable paths starting in $c$. We say a path $\gamma$ dominates a path $\delta$ conditional on $Y$ if there is a common node $a$ and a coalition $S_a$ that is effective in bringing about $\gamma$ and weakly prefers to $\delta$ all stable continuations of $\gamma$ starting in $a$’s successor on $\gamma$, $c$.

**Definition 7.** Let $a$ be a node common to $\gamma$ and $\delta$ and $c \in \gamma$, $c \notin \delta$, the successor of $a$ on $\gamma$. A path $\gamma \in X(a)$ dominates path $\delta \in X(a)$ via $\triangleleft^Y$ conditional on $Y \subseteq X(c)$, i.e. $\delta \triangleleft^Y \gamma$, if there is $a \xrightarrow{S} c$ and for all $\beta \in Y$: $\beta \succeq_S \delta$.

Using this dominance relationship, we define the stable set $V(X, \triangleleft^Y)$. Formally, this definition is closely related to CSSB of a path situation (see Xue [29]): in CSSB a path $\delta$ is blocked if there is a deviation from $\delta$ to a node $c \notin \delta$ such that the deviating coalition strictly prefers to $\delta$ all conservatively stable paths starting in $c$. In our definition, blocking depends on a weak preference relation. As we are going to show, the union of weakly conservatively stable sets of paths $\Sigma V(X, \triangleleft^Y) =$ outcomes for all best responses of the other players to $x'$, $y \in Y(v)$, in the subgame starting in some node $v$ are strictly preferred by the decision maker. Based on generalized stable set, Huang and Luo [13] define sequentially stable equilibrium SSE: in our example $\{\alpha, \gamma\}$ is an SSE because it does not pay for $S_a$ to switch to $\gamma$ for all strategic choices of $C$.

\textsuperscript{13}In a departure from standard assumptions on expectations, Bloch and Van Den Nouweland [4] introduce heterogenous expectations in modeling farsighted stability.
∪ₖ𝑉^k(X, ⪯^Y) refines the union of weakly stable sets of paths, Σ𝑉(X, ⪯) and, hence, CSSB. Moreover, if the game has a representation as a simple tree, the union of conservatively stable sets of paths coincides with SEG.

Again, 𝑉(X, ⪯^Y) is a stable set if it is internally and externally stable for the dominance relation ⪯^Y. For any node a and successor node c we define Y the union of stable sets in the remainder of the game starting in c.

**Definition 8.** For a = vₜ with t < T let c be a successor node c ∈ Ω(a). Define Y = c if t = T − 1 and Y ⊆ X(c) : Y = ∪ₖ𝑉^k(X(c), ⪯^Y), else. V is a weakly conservative stable set of paths if for all a ∈ Z (X(a), {⪯^Y}) V is internally and externally stable conditional on Y: It is internally stable if α ∈ V(X(a), ⪯^Y) implies that there is no α ∈ V(X(a), ⪯^Y) such that α ⪯^Y β. And it is externally stable if for all γ ∈ X(a) \ V(X(a), ⪯^Y) there is α ∈ V(X(a), ⪯^Y) such that γ ⪯^Y α.

WCSSP solves the problem of inclusion of weakly dominated paths (in the conventional, strategic sense) in WSSP. Consider again example 2 b) where C is indifferent between γ and δ and Sₐ has the preference order γ ⪰ₐ α ≃ₐ δ. There are two weakly conservative stable sets for the continuation of the game on X(c) and Y = {γ, δ}. γ and δ weakly dominate α via ⪯^Y, hence WCSSP is unique up to the choice of C with V¹(X, ⪯^Y) = {γ} and V²(X, ⪯^Y) = {δ}.

### 3.1 Existence of WCSSP

Existence of weakly conservative stable set can be shown following the same line of argument as in lemma 1 and proposition 1:

**Proposition 5.** Assume that ⪯^Y is acyclic at all z ∈ Z. Then a weakly conservative stable set of paths for (X, {⪯^Y}, Y ⊆ X) exists.

Note that Y = ∪ₖ𝑉^k itself is not stable, i.e. it generally lacks internal stability.

### 3.2 Relationship between WCSSP and SGE

While WCSSP solves the problem of inclusion of weakly dominated paths (in the conventional, strategic sense) in WSSP, the one-to-one relationship between perfect equilibrium paths and stable set that characterizes WSSP is lost.

Consider the generic game in figure 4. Player B moves at node v₁ or v₂ if that node is reached. At each node, she is indifferent between her choices of γ and δ or β and α, respectively. Player A moves in node v₀ where she has a choice between α and γ (for convenience we suppress the paths δ and β, the truncations of which continue in nodes v₁ and v₂).
Example 6. Assume, A’s preferences are as follows: $\alpha \succ_A \gamma \succ_A \beta \succ_A \delta$. In this case, the set of subgame perfect Nash equilibria is $SGE = \{\{\gamma, \beta\}, \{\alpha, \gamma\}, \{\alpha, \delta\}, \{\beta, \delta\}\}$.

At $v_1^1$, we have $Y_1^1 = \{\gamma, \delta\}$ and at $v_2^1$ we have $Y_1^2 = \{\beta, \alpha\}$. The dominance relations $\delta \triangleleft Y \alpha, \delta \triangleleft Y \beta$, but $\gamma \not\triangleleft Y \alpha, \gamma \not\triangleleft Y \beta$ (and vice versa). We obtain the weakly conservative stable sets: $V^1(X, \triangleleft Y) = \{\alpha, \gamma\}, V^2(X, \triangleleft Y) = \{\gamma, \beta\}, V^3(X, \triangleleft Y) = \{\alpha, \delta\}, V^4(X, \triangleleft Y) = \{\beta, \delta\}$.

Note that the union of all $V(X, \triangleleft Y)$ comprises all paths that may be played as part of a subgame perfect Nash equilibrium. $\beta$ and $\gamma$ are both included in one stable set as neither path conservatively dominates the other, although $A$ would want to deviate from $\beta$ if she were presented with a direct choice between $\beta$ and $\gamma$. The test for subgame perfect equilibrium assumes that players face this kind of choice while the decision situation of $A$ is such that she might choose path $\alpha$ in the hope that $B$ chooses $\alpha$ rather than $\beta$.\(^{14}\)

Example 7. Assume $\alpha \succ_A \beta \sim_A \gamma \succ_A \delta$. In this case, the set of subgame perfect Nash equilibria is $SGE = \{\{\alpha, \gamma\}, \{\alpha, \delta\}, \{\beta, \gamma\}, \{\beta, \delta\}\}$.

\(^{14}\)CSSB and WCSSP agree on the decision-theoretic assessment of the situation.
We have $Y_1^1 = \{\gamma, \delta\}$, $Y_2^2 = \{\beta, \alpha\}$ and the dominance relations are $\gamma \triangleleft \alpha$, $\alpha (\beta) \triangleleft \gamma$ and $\alpha (\beta) \triangleleft \gamma$. So the conservative stable sets are $V^1 (X, \triangleleft) = \{\alpha, \gamma\}$, $V^2 (X, \triangleleft) = \{\alpha, \delta\}$, $V^3 (X, \triangleleft) = \{\beta, \gamma\}$, $V^4 (X, \triangleleft) = \{\beta, \delta\}$.

Note that $\gamma$ is excluded from $SGE$ and $\Sigma V (X, \triangleleft)$ but $\{\gamma, \beta\}$ is a WSSP.

**Example 8.** Assume $\alpha \sim_A \beta \sim_A \gamma \sim_A \delta$. In this case, the set of subgame perfect Nash equilibria is $SGE = \{\{\alpha, \gamma\}, \{\alpha, \delta\}, \{\beta, \gamma\}, \{\beta, \delta\}, \{\gamma, \alpha\}, \{\gamma, \beta\}, \{\delta, \alpha\}, \{\delta, \beta\}\}$.

Now, $Y_1^1 = \{\gamma, \delta\}$ and $Y_2^2 = \{\beta, \alpha\}$. We have the dominance relations $j \triangleleft l$ and $l \triangleleft j$ for $j \in \{\alpha, \beta\}$, $l \in \{\gamma, \delta\}$. The weakly conservative stable sets are $V^1 (X, \triangleleft) = \{\alpha, \gamma\}$, $V^2 (X, \triangleleft) = \{\alpha, \delta\}$, $V^3 (X, \triangleleft) = \{\beta, \gamma\}$, $V^4 (X, \triangleleft) = \{\beta, \delta\}$, $V^5 (X, \triangleleft) = \{\gamma, \alpha\}$, $V^6 (X, \triangleleft) = \{\gamma, \beta\}$, $V^7 (X, \triangleleft) = \{\delta, \alpha\}$, $V^8 (X, \triangleleft) = \{\delta, \beta\}$.

In examples 7 and 8, each weakly conservative stable set corresponds to an element of $SGE$, although this is not always the case (as in example 6). In general, for games that can be represented as simple trees, the set $\cup_k V (X, \triangleleft)$ coincides with the set $SGE$.

**Proposition 6.** Consider a game with a representation as a simple tree: there is $V (X, \{\triangleleft\})$ such that path $\alpha \in V (X, \{\triangleleft\})$ if and only if $\alpha \in SGE$.

The union of all paths in WCSSP, $\Sigma V (\triangleleft) = \cup_j V (\triangleleft)$, collects all paths in $SGE$ which may be played as part of a subgame perfect Nash equilibrium. But while example 6 above shows that there is no one-to-one relationship between subgame perfect equilibrium and conservative stable set, each combination of paths corresponding to any particular subgame perfect Nash equilibrium may be retrieved by computing all WSSP for the set $\Sigma V (\triangleleft)$.

### 3.3 Relationship between WCSSP, WSSP and OSSB

Finally, we relate WCSSP to the remaining concepts in this paper. We already have shown (see proposition 3) that WSSP refines CSSB. For the case of simple trees, we have shown\(^{15}\) that $V (\triangleleft)$ refines $V (\triangleleft)$ by excluding paths that are dominated in the strategic sense. But because WCSSP, like CSSB, blocks conservatively while WSSP blocks optimistically, we can show that in the case where the solution is multivalued at some node (as in example 3 b) there are paths in $\Sigma V (\triangleleft)$ which are not in $\Sigma V (\triangleleft)$. The following propositions summarizes the latter two results:

\(^{15}\)See propositions 4 and 6.
Figure 5: Relationship between solution concepts in the case where WSSP and WCSSP exist and OSSB and CSSB assign a non empty solution.

**Proposition 7.** Assume that $\triangle$ and $\triangle^Y$ are acyclic at all $z \in Z$. WCSSP is neither includes nor is included in WSSP, i.e. $\Sigma V(X, \triangle^Y) \not\subseteq \Sigma V(X, \triangle) \text{ and } \Sigma V(X, \triangle) \not\subseteq \Sigma V(X, \triangle^Y)$.

Moreover, while WCSSP and OSSB may overlap, neither set generally includes the other.

**Proposition 8.** Assume that $\triangle$ and $\triangle^Y$ are acyclic at all $z \in Z$. WCSSP neither includes nor is included in OSSB.

Figure 5 illustrates the relationship for the case where the stable set of paths exists and CSSB and OSSB assign non empty solutions at each node.

4 Extensions

4.1 Unbounded games

Consider the example of an infinite game due to Greenberg [10]: players one and two move sequentially with player one calling $u$ or $d$ and player two calling $l$ or $r$. 

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After an infinite sequence where \( u, d, l \) and \( r \) each have been called infinitely often, player one receives 5 and player two receives 0. After a sequence where at least one of the choices \( u, d, l \) or \( r \) have only been called finitely often, player one receives zero and player two receives 5. Player two can ensure 5 by choosing only \( l \) and never \( r \) whenever it is her turn, so SGE predicts a payoff of \((0, 5)\). There are, however, two CSSB, one where players receive \((0, 5)\) and one where players receive \((5, 0)\).

Weakly stable set of paths \( V(X, \triangleleft) \) for this situation agrees with SGE and excludes paths with infinite sequences \( u, d, l \) and \( r \): Consider a node where agent one moves. Path \( \gamma \) starts with picking \( l \) and \( r \) is never picked. Path \( \beta \) starts with picking \( r \) and along \( \beta \) all choices are picked infinitely often. Then \( \beta \triangleleft \gamma \) and \( \gamma \not\triangleleft \beta \).

Using weakly conservative stable set of paths, \( V(X, \triangleleft^Y) \) and applying definition 7, \( \gamma \not\triangleleft^Y \beta \) because not all possible paths after an initial move in the direction of \( \beta \) are at least as good as \( \gamma \). On the other hand, \( \beta \triangleleft^Y \gamma \) because all possible paths after an initial move in the direction of \( \gamma \) are at least as good as \( \beta \). Note that if we had instead defined \( \triangleleft^Y \) with a strong preference relation, the conservative stable set of paths would also include paths with infinite sequences \( u, d, l \) and \( r \).

4.2 Cyclic Graphs

For dynamic games as considered in this paper we can naturally exclude cyclic graphs. However, we may apply the definition of stable set to an environment that permits cyclic graphs. Consider the cyclic graph in figure 6. Note that OSSB does not exist (see Pech, 2017) but CSSB does (see Xue, 1998).

For this example, we cannot construct WSSP. The argument parallels the demonstration of non existence of OSSB: suppose that \( v_2 \in V(v_2, \triangleleft) \), hence, \( \beta \in V(v_1, \triangleleft) \). But then \( \alpha \not\in V(v_3, \triangleleft) \) by internal stability and \( \gamma \in V(v_2, \triangleleft) \) by external stability. From this follows \( v_2 \triangleleft \gamma \) contradicting that \( v_2 \in V(v_2, \triangleleft) \).

Now consider WCSSP: assume that \( \gamma \) and \( v_2 \in Y(v_2) \). Then \( v_1 \not\triangleleft^Y \beta \) because not all \( \delta \in Y(v_2) \) satisfy \( \delta \succ^C v_1 \). The same construction can be repeated for all nodes, so that the unique WCSSP consists of the nodes \( \{v_1, v_2, v_3\} \) which coincides with CSSB. Thus, when applied to cyclic graphs existence of WCSSP closely mirrors existence of CSSB.

4.3 Weakly Socially Stable Set of Paths WSSSP

Making non cyclicity of \( \triangleleft \) a precondition for the existence of a solution is undesirable. As we have argued in the discussion of example 1, a more appropriate way of dealing with cycles of odd length is to say that any element of the cycle may be selected
and, hence, to include the cycle in the solution. Following Delver and Monsuur’s (2001) definition of socially stable set we say that two elements may be selected in a solution if one element indirectly dominates the other and this relationship is “balanced” by the latter element indirectly dominating the former. To this end, we define the transitive closure of the dominance relation $\triangleleft$:

**Definition 9.** Define the transitive closure of $\triangleleft$ conditional on $W$: $\alpha_1 \triangleleft_{W} \alpha_K$ if there is a sequence on $\{\alpha_1, ..., \alpha_K\} \subseteq W$ such that $\alpha_k \triangleleft \alpha_{k+1}$, $k = 1, ..., K - 1$.

Using transitive closure of $\triangleleft$ we generalize weakly stable set of paths:

**Definition 10.** For every node $a \in Z$, the set $V(X(a), \triangleleft_{W}) \subseteq X(a)$ is weakly internally socially stable if $\alpha \triangleleft_{W} \beta$ implies $\beta \triangleleft_{W} \alpha$. And it is weakly externally socially stable if for all $\gamma \in X(a) \setminus V(X(a), \triangleleft_{W})$ there is $\alpha \in V(X(a), \triangleleft_{W})$ such that $\gamma \triangleleft \alpha$. The weakly socially stable set of paths, $V(X, \triangleleft_{W})$ assigns $V(X(a), \triangleleft_{W})$ to all all non terminal nodes $a \in Z$ such that $V(X, \triangleleft_{W})$ is weakly internally and weakly externally socially stable.

Because socially stable set always exists, establishing existence of WSSSP is straightforward:

![Cyclic Graph](image-url)

Socially stable set is typically non unique. For example, for \( X = \{\alpha, \beta\} \) with the weak dominance relationships \( \alpha \bowrightarrow \beta, \beta \bowrightarrow \alpha \), there are three weakly socially stable sets: \( V^1(\bowrightarrow) = \{\alpha\}, V^2(\bowrightarrow) = \{\beta\}, V^3(\bowrightarrow) = \{\alpha, \beta\} \) rather than two stable sets as is the case with \( V(\subset) \).

So while there is obviously no one-to-one relationship between subgame perfect path and weakly socially stable set of paths, we may still wish to relate the union of weakly socially stable sets \( \Sigma V(\bowrightarrow) \) to the set of subgame perfect paths. First, note that \( \Sigma V(\bowrightarrow) \) is not itself socially stable: assume \( \alpha \not\bowrightarrow \beta \) and \( \beta \not\bowrightarrow \alpha \) but each is dominated by another alternative, i.e. \( \alpha \bowrightarrow \gamma \) and \( \beta \bowrightarrow \delta \) and we also have the pair of relations \( \gamma \bowrightarrow \delta \) and \( \delta \bowrightarrow \gamma \) for which we write \( \gamma \leftrightarrow \delta \) to simplify notation. The weakly socially stable sets of paths are \( V^1(\bowrightarrow) = \{\gamma, \beta\}, V^2(\bowrightarrow) = \{\gamma, \delta\} \) and \( V^3(\bowrightarrow) = \{\gamma, \delta\} \). Yet the union of socially stable sets is \( \{\alpha, \beta, \gamma, \delta\} \) which is not socially stable.

WSSSP addresses the problem of non existence of WSSP in the presence of cycles. Yet as the following example shows, it may "overgeneralizes" when the graph is acyclic. Recall that by definition 4, relationships of the form \( \alpha \leftrightarrow \beta \) in a decision loop break the cycle.

Example 9. Assume the following relationships exist between paths in \( X \): \( \alpha \leftrightarrow \beta \) and \( \alpha \leftrightarrow \gamma \) but \( \gamma \bowrightarrow \beta \).

As \( \alpha \leftrightarrow \beta \), if \( \alpha \) or \( \beta \) are included in some \( V(\subset) \), only one \( \subset \)-relation may be active, so either \( \alpha \) or \( \beta \) is included in any one stable set. In this case, \( V^1(\subset) = \{\alpha\} \) and \( V^2(\subset) = \{\beta\} \). Yet \( \{\alpha, \beta, \gamma\} \) is a weakly socially stable set \( V(\bowrightarrow) \) which shows that not all weakly socially stable paths are also weakly stable. As this example shows, acyclicity of \( \subset \) is not a sufficient condition for the union of \( V(\bowrightarrow) \) and the union of \( V(\subset) \) to coincide. However, if \( \subset \) establishes a complete order on \( X(z) \), this ensures such an outcome:

Proposition 10. Assume that at each \( z \), \( \subset \) establishes a complete order on \( X(z) \). Then \( \alpha \in \Sigma V(\bowrightarrow) \) if and only if \( \alpha \in \Sigma V(\subset) \).
5 Conclusion

In this paper we have proposed weakly stable set of paths (WSSP) and weakly conservative stable set of paths (WCSSP) as solution concepts for dynamic coalitional games which can be represented as trees. The two solution concepts are related but different from CSSB and OSSB, two standards of behavior (SB) established in the Theory of Social Situations, as figure 5 in the text shows. Our dominance relationship - \(<\) in the case of WSSP and \(<^y\) in the case of WCSSP - requires that deviating agents weakly prefer the path to which they deviate. WSSP is optimistic in the sense that agents deviate when they have the possibility of being weakly better off as a result and WCSSP is conservative in the sense that agents only deviate when all stable continuations after a deviation leave them weakly better off. In the case of simple trees - game trees with single player moves - the set of all weakly stable paths, as is the case with CSSB, coincides with the set of perfect equilibrium paths and the set of all weakly conservative stable paths coincides with the set of paths that yield subgame perfect Nash equilibrium, i.e., that are iteratively undominated in the strategic sense. Stable set and the Theory of Social Situations fail to assign a satisfactory solution in the case of decision cycles - which leads to non existence in the case of SSP and assignment of the empty set in the case of SB. We solve this problem by applying socially stable set (Delver and Monsuur, [6]), which always exists, to generalize our solution. Accordingly, we define weakly socially stable set of paths (WSSSP) which, like WSSP, generalizes the set of perfect equilibrium paths.

6 Appendix: Proofs

Proof of lemma 1  Let \(X^0\) be the set of paths starting node in \(v_{T-1}\) at the last stage of the game.

Because by assumption \(<\) is acyclic on \(X^0\) and \(X(v_{T-1})\) is finite, there exists at least one maximal element with respect to \(<\), that is an element \(\alpha\) such that any relation \(\alpha < \beta\) is countered by the relation \(\beta < \alpha\). Let \(M_X^0\) be the set of maximal elements. Select \(\beta \in M_X^0\) and let \(\widehat{M}_X^0 = \{ \beta \in M_X^0 : \beta \not< \alpha, \alpha \in \widehat{M}_X^0 \}, \text{dom}(\widehat{M}_X^0) = \{ \alpha \in W^0 : \alpha < \beta, \beta \in \widehat{M}_X^0 \}\) and \(X^1 = X^0 \setminus (\widehat{M}_X^0 \cup \text{dom}(\widehat{M}_X^0))\).

If \(X^1 = \emptyset\) we are done. Otherwise, let \(M_X^1\) be the set of maximal elements with respect to \(<\) in \(X^1\). Select \(\gamma \in M_X^1\) and let \(\widehat{M}_X^1 = \{ \gamma \in M_X^1 : \gamma \not< \delta, \delta \in \widehat{M}_X^1 \}\). Note that \(\widehat{M}_X^0 \cup \widehat{M}_X^1\) satisfies the internal stability criterion: Suppose that \(\gamma \in \widehat{M}_X^1\) and \(\beta < \gamma\). But \(\beta\) is maximal in \(X^0\) and if \(\beta < \gamma\) and \(\gamma < \beta\) then in the constructed solution, \(\gamma \in \text{dom}(\widehat{M}_X^0)\) and, therefore, \(\gamma \not\in \widehat{M}_X^1\). Hence, either \(\beta\) is not maximal or
Proof of proposition 1 Consider node $v'_{t-1}$ and assume that for all successor nodes $v^k_t \in \Omega(v_{t-1})$ we have assigned $V(v^k_t)$ and let $\alpha(v_t) \in V(v^k_t)$.

Then, by external stability, $\alpha(v'_{t-1}) \in V(v'_{t-1})$ unless there is $\gamma \triangleright v'_{t-1}$ and $\alpha(v'_{t-1}) \triangleright v_{t-2}$. Using the steps in the proof of lemma 1 we can show that provided all $V(v^k_t)$ exist, $V(v'_{t-1})$ exists. As we have demonstrated this result in lemma 1 for $t = T - 1$, it holds by induction for all $t < T - 1$.

Proof of proposition 2 1) Show that $\alpha \in \sigma^\text{OSSB}$ implies $\alpha \in \Sigma V(\gamma)$.

$\alpha \in \sigma^\text{OSSB}(v_{t-1}) \Rightarrow [\text{There is no } \beta \in X(v_{t-1}) \text{ and effective coalition } S_{t-1} \text{ in } v_{t-1} \text{ with } \beta_{v_{t-1}} \in \sigma(v_{t}) \text{ such that } \beta \triangleright S_{t-1}, \alpha] \Rightarrow [\text{either: (a) } \alpha \sim S_{t-1}, \beta \text{ or (b) } \alpha \triangleright S_{t-1}, \beta] \Leftrightarrow \beta \triangleright \alpha$.

Assume that for all $s \geq t - 1$: $\sigma^\text{OSSB}(v_s) \subseteq \Sigma V(v_s, \triangleleft)$. Consider a predecessor node $v_{t-2}$ and suppose there is $\pi$ such that $\pi \in \sigma^\text{OSSB}(v_{t-2})$ but $\pi \not\in V(v_{t-2}, \triangleleft)$.

For this we need to have $\delta \not\in \sigma^\text{OSSB}(v_{t-1})$, $\delta \in V(v_{t-1}, \triangleleft)$ such that $\pi \triangleright \delta$.

$\delta \not\in \sigma^\text{OSSB}(v_{t-1}) \Leftrightarrow [\text{there is } \gamma \text{ and } S_t \text{ such that } \gamma \triangleright S_t, \delta]$. Because $\delta \in V(v_{t-1}, \triangleleft)$, either $\gamma \triangleright \delta$ or there is $\eta \in V$ such that $\gamma \triangleright \eta, \delta \not\triangleleft \eta$.

In both cases, there is $V(v_{t-2}, \triangleleft)$ such that $\pi \in V(v_{t-2}, \triangleleft)$, contradicting the supposition.

Hence, $\sigma^\text{OSSB}(v_s) \subseteq \Sigma V(v_s, \triangleleft)$ for $s \geq t - 1$ implies $\sigma^\text{OSSB}(v_{t-2}) \subseteq \Sigma V(v_{t-2}, \triangleleft)$.

The relationship $\sigma^\text{OSSB}(v_t) \subseteq \Sigma V(v_t, \triangleleft)$ clearly holds for a terminal node, i.e. for $v_t = v_T$, for which $\alpha \in \sigma^\text{OSSB}$ implies $\alpha \in V(v_T, \triangleleft)$ and, for node $v_{T-1}$ at the last stage of the game we can induce $\sigma^\text{OSSB}(v_{T-1}) \subseteq \Sigma V(v_{T-1}, \triangleleft)$. Then, it must also hold for any predecessor node. This concludes the induction.

2) Example 5 in the text shows that the reverse is not true.

Proof of proposition 3 Before establishing the proposition, we show that the following intermediate claim is true:

Claim: Assume that for all $v \in Z$, $\triangleleft$ establishes a complete order on $X(v)$. Then $\gamma, \alpha \in \sigma^\text{CSSB}(a)$ and $\alpha \triangleright \gamma$ implies that (a) $\gamma \triangleright \alpha$ or (b) there exists $c \in \gamma$ and $\phi \in \sigma^\text{CSSB}(c)$ such that $\gamma \triangleright \phi$ and $\phi \triangleright \alpha$.

Note that the last two steps do not claim stability of $\beta$ but this statement rather says that even if $\beta_{|v_t} \in V(v_t, \triangleleft)$, $\alpha \in V(v_{t-1}, \triangleleft)$.
Proof. First, consider part (a) of the claim: $\alpha \triangleleft \gamma$ and $\gamma \triangleleft \alpha$ imply that for the decisive coalition $S_a$, $\alpha \not\triangleright_{S_a} \gamma$ and $\gamma \not\triangleright_{S_a} \alpha$ and $\gamma, \alpha \in \sigma^{CSSB}(a)$. By completeness, $\alpha \sim_{S_a} \gamma$.

To see that conditions (a) and (b) are exhaustive, suppose that neither (a) nor (b) holds. In particular, there is no node $c \in \gamma$ and $\phi \in \sigma^{CSSB}(c)$, such that $\gamma \triangleleft \phi$ and (in node $a$) $\phi \triangleleft \alpha$. If there is no $\phi$ that blocks $\gamma$, then $\gamma$ is the unique stable continuation following a deviation from $\alpha$ in $a$ and, because $\alpha \triangleleft \gamma$ and $\gamma \not\triangleleft \alpha$, we must have $\gamma \succ_{S_a} \alpha$, contradicting that $\alpha \in \sigma^{CSSB}(a)$. Next, suppose that there is $\phi \in \sigma^{CSSB}(c)$ and $\gamma \triangleleft \phi$ but $\phi \not\triangleleft \alpha$. In this case, $\phi \succ_{S_a} \alpha$, and all stable continuations of the deviation in $a$ from $\alpha$ are strictly preferred by $S_a$ to $\alpha$, contradicting that $\alpha \in \sigma^{CSSB}(a)$.

This establishes the claim.

**Part a)** Assume that $\triangleleft$ establishes a complete order. Then $\alpha \in \sigma^{CSSB}$ implies that there is $V(\triangleleft)$ such that $\alpha \in V(\triangleleft)$:

- $\alpha \in \sigma^{CSSB} \iff$ [there does not exist a deviation to $c \in \gamma$, $c \not\in \alpha$, such that for all potential continuations from $c$, $\gamma$, $\phi \in \sigma^{CSSB}(c)$, the coalition effective in the deviation, $S_\gamma$, has preferences $\gamma \succ_{S_\gamma} \alpha$ and $\phi \succ_{S_\gamma} \alpha$].
- $\iff$ at least one: $[\gamma \not\triangleright_{S_\gamma} \alpha]$ or $[\phi \not\triangleright_{S_\gamma} \alpha]$.
- $\iff$ One of the following must be true:
  - (aa) $\alpha \not\triangleleft \gamma$ and $\alpha \not\triangleleft \phi$;
  - (ab) $\alpha \triangleleft \gamma$ and $\gamma \triangleleft \alpha$ (or the corresponding condition for $\phi$);
  - (ac) $\alpha \triangleleft \gamma$ and there is $\phi \in \sigma^{CSSB}(c)$ such that $\gamma \triangleleft \phi$ and $\phi \triangleleft \alpha$ (or the corresponding condition with $\gamma$ and $\phi$ switching places).

The last step follows from completeness of $\triangleleft$. In case (aa) and (ab), we can construct $V(X, \triangleleft)$ with $\alpha \in V(X, \triangleleft)$. In the case of (ac), by the claim, we also have $\gamma \triangleleft \phi$, so we can construct $V(\triangleleft)$ with $\alpha$, $\phi \in V(\triangleleft)$.

Hence, in the case where $\triangleleft$ is complete, $\alpha \in \sigma^{CSSB}(a)$ implies $\alpha \in \Sigma V(a, \triangleleft)$, or $\sigma^{CSSB}(a) \subseteq \Sigma V(a, \triangleleft)$.

**Part b)** Assume that the order $\triangleleft$ is not complete. Then $\alpha \in \sigma^{CSSB}$ does not imply that there is $V(\triangleleft)$ such that $\alpha \in V(\triangleleft)$:

When $\triangleleft$ is not complete, in any node we may have $\phi \not\triangleleft \gamma$ and $\gamma \not\triangleleft \phi$. Say, this holds for all paths emanating in node $c$, such as in example 3. Then also $\phi \not\triangleleft \gamma$ and $\gamma \not\triangleright \phi$ and we have $\gamma, \phi \in \sigma^{CSSB}(c)$. As $\gamma, \phi \in V(c, \triangleleft)$ follows directly, CSSB and WSSP make the same prediction.

Now consider node $a$ which is prior to node $c$ and coalition $S_a$ deciding between $\alpha$ and $\gamma$. If $\triangleleft$ does not rank $\alpha$ and $\gamma$, CSSB and WSSP make the same prediction, so assume that the order $\triangleleft$ is complete in $a$ (i.e., it is only incomplete in $c$).

$\alpha \in \sigma^{CSSB}$ if at most one: $\gamma \succ_{S_a} \alpha$, $\phi \succ_{S_a} \alpha$. 

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\(\alpha \in V(a, \prec)\) if neither \([\alpha \prec \gamma\) and \(\gamma \not\prec \alpha]\) nor \([\alpha \prec \phi\) and \(\phi \not\prec \alpha]\), i.e. neither \(\gamma \succ_a \alpha\) nor \(\phi \succ_a \alpha\). Hence, \(\alpha \in \sigma^{CSSB}\) but \(\alpha \not\in \Sigma V(a, \prec)\).

As we have shown, in the case where \(\prec\) is not complete, \(\alpha \in \sigma^{CSSB}(a)\) does not imply \(\alpha \in \Sigma V(a, \prec)\) and, therefore, \(\sigma^{CSSB}(a) \not\subseteq \Sigma V(a, \prec)\).

\textbf{Part c)} Show that \(\alpha \in \Sigma V(a, \prec)\) implies \(\alpha \in \sigma^{CSSB}\).

Let \(\alpha \in V(X(a), \prec)\) and \(\gamma, \delta \in \Sigma V(X(c), \prec)\) where \(c\) is a successor node of \(a\), \(c \not\in a\). Assume \(\alpha \in \Sigma V(a, \prec)\).

\(\Rightarrow\) for all \(c \in \Omega(a), c \not\in a\) one of the following conditions has to hold:
1. \((ba)\) if for some \(\alpha \not\in \gamma\) \(\Rightarrow \gamma \not\prec \alpha \Rightarrow \alpha \in \sigma^{CSSB}\).
2. \((bb)\) if for some \(V^j, \gamma \in V^j(X(c), \prec)\): \(\alpha \prec \gamma\), then \([\gamma \prec a]\) or \([\text{there is } \delta \in \Sigma V(X(c), \prec), \alpha \not\in a\) and, hence, \(\alpha \in \sigma^{CSSB}(a)\). In the second case, \(\delta \not\prec_a \alpha\) and, hence, \(\alpha \in \sigma^{CSSB}(a)\). This shows that \(\Sigma V(a, \prec) \subseteq \sigma^{CSSB}(a)\).

\textbf{Proof of proposition 4} We have to show that \(\alpha \in SGE(v_t)\) implies that there is \(V(X(v_t), \prec)\) such that \(\alpha \in V(X(v_t), \prec)\):

At the last stage, \(T - 1\), let \(\alpha \in SGE(X(v_{T-1}))\) and \(\succ_{T-1}\) be the complete preference order of the decision maker in \(T - 1\).

\begin{align*}
\Leftrightarrow & \, \alpha \succ_{T-1} \beta, \forall \beta \in X(v_{T-1}) \\
\Leftrightarrow & \, \beta \not\prec \alpha, \forall \beta \in X(v_{T-1}) \\
\Leftrightarrow & \, \exists V^k(v_{T-1}) \text{ with } \alpha \in V(v_{T-1}).
\end{align*}

Now consider any previous stage \(t - 1\) and node \(v_{t-1}\) and assume that for all successor nodes \(b, \beta, b \in SGE(b)\) implies that there is \(V(b, \prec)\) with \(b \in V(b, \prec)\).

Let \(\alpha \in SGE(v_{t-1}), a_t \in \alpha\) and \(a_t \in \Omega(v_{t-1})\).

\(\Rightarrow\) for all successors \(\gamma \in \Omega(v_{t-1})\{a_t\}\): there exists \(\gamma_{ct} \in SGE(c_t)\) such that \(\alpha_{ct} \succ_{t-1} \gamma_{ct}\)

\(\Leftrightarrow\) for all \(\gamma \in X(v_{t-1})\{a_t\}\) and \(\gamma \in X(v_{t-1})\) with \(a_t \in \gamma\): there is at least one path \(\gamma'\) with continuation \(\gamma'_{ct} \in \Sigma V(c_t, \prec)\) such that \(\gamma' \not\prec \alpha\)

\(\Leftrightarrow\) \(\exists V(v_{t-1}, \prec)\) with \(\alpha \in V(v_{t-1}, \prec)\). This establishes the proposition.

Note that from the last two steps it follows that if \(\gamma \not\prec \alpha\) and \(\alpha, \gamma \in SEG(v_{t-1})\) then \(\alpha \prec \gamma\) and \(\gamma \prec \alpha\) and, hence \(\alpha\) and \(\gamma\) are not in the same stable set.

A subgame perfect strategy profile consists of actions \(s_{v_k}\) at node \(v_k^i\) such that at each node and each alternative action \(s_{v_k}\) that induces an equilibrium path \(\gamma_{v_k^i}\) starting in a successor of \(v_k^i\), \(s_{v_k} \succ \gamma_{v_k^i}\). Therefore, if \(\gamma'_{ct}\) and \(\alpha\) are part of a subgame perfect strategy profile, then \(\alpha_{ct} \succ_{t-1} \gamma'_{ct}\) and \(\alpha, \gamma'_{ct} \in V(X, \prec)\). Suppose that \(\gamma'_{ct} \succ_{t-1} \alpha_{ct}\). Then \(\alpha\) and \(\gamma'_{ct}\) are not part of a subgame perfect strategy profile and \(\alpha \not\prec \gamma', \gamma' \not\prec \alpha\), contradicting that \(\alpha \in V(X, \prec)\).
Proof of proposition 5  Applying the argument in lemma 1, at the last stage there is at least one stable set $V(X(v_{T-1}), α^Y)$ for any $v_{T-1}$, $V(X(v_{T-1}), α^Y)$ satisfies the internal stability criterion and, hence, $Y$ is non empty. Moreover, for all $δ \notin V(X(v_{T-1}), α^Y)$, there is $α \in V$ with $δ α^Y$ $α$, so $V(X(v_{T-1}), α^Y)$ is externally stable. Repeating this argument for stages $T - 2$, $T - 3$, ..., 1 proves the proposition.

Proof of proposition 6  Show that $α \in SGE$ if and only if there is $V(α^Y)$ with $α \in V(α^Y)$.

Consider node $v_{t-1}$ with successors $a \in α$ and $c \in γ$, $c \notin α$ and assume that $Y(a) = SGE(a)$ and $Y(c) = SGE(c)$.

a) "if" part: $α \in SGE$ if $α \in V^K(α^Y)$

$α$ is included in some $V(X, α^Y)$ if it is either unblocked by $α^Y$ (case aa) or, if it is blocked via $α^Y$ for some path in $Y(c)$, then there is another conservatively stable set of paths in which $α$ blocks via $α^Y$ all paths in $Y(c)$ (case ab):

case (aa)

$\forall c \in Ω(v_0), c \neq a, \forall γ \in Y(c): α \nabla^Y γ$

$⇔$ not: $[\forall δ \in Y(c) : δ \succeq_A α]$.

Hence, $α$ is not dominated in the conventional (strategic) sense and $α \in SGE$.

case (ab)

If there is $γ \in Y(c)$ such that $α \nabla^Y γ$, then $∀δ \in Y(c): δ \nabla^Y α$.$^{17}$

Hence, $α \in SGE$. The left-hand part of figure 7 illustrates this case with $β$ a typical path in $Y(a)$ and paths assigned to each node ordered by their preference ranking.$^{18}$

To see that cases (aa) and (ab) exhaust all possibilities for $α$ to be included in some $V(α^Y)$, we have to show that it is impossible for another node $h$ and path $η \in Y(h)$ to indirectly remove any block from $α$ by dominating any blocking path $γ$ or $δ$:

Suppose $α \in V(α^Y)$ and neither case (aa) nor (ab) applies. Then there must be a node $h \in Ω(v_{t-1})$ and $η \in Y(h)$ such that for all $δ \in Y(c)$, $δ \nabla^Y h$ $η$ and $α \nabla^Y h$. By transitivity, all $η, φ \in Y(h)$ satisfy $φ \succeq_A α$, contradicting $α \nabla^Y h$ $η$ (see the right-hand part of figure 7). This shows that $α \in V(α^Y)$ only if either case (aa) or (ab) applies.

b) "only if" part: $α \in SGE$ only if $α \in V^K(α^Y)$:

$^{17}$Note that if $α \nabla^Y γ$, it follows that $α \nabla^Y δ$ for all $δ \in Y(c)$. Hence, the reverse condition needs to hold for all $δ \in Y(c)$ including $γ$.

$^{18}$Note that all paths in $Y(c)$ are weakly preferred to $α$, but they are not necessarily weakly preferred to $β$. 

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\( \alpha \not\in V^k(\preceq^Y) \Rightarrow \) there exists \( \gamma \in V^k(\preceq^Y) \) with \( \alpha \preceq^Y \gamma \) and \( \gamma \not\succeq^Y \alpha \).\(^{19}\) Show that \( \alpha \) is dominated in \( v_{t-1} \) in the conventional (strategic) sense. We have

\[
\gamma \succeq_A \alpha \land \forall \delta \in Y(a) : \delta \succeq_A \alpha; \quad (1)
\]

and not: \([\alpha \succeq_A \gamma \land \forall \beta \in Y(a) : \beta \succeq_A \gamma]\). \(^{(2)}\)

From from (2) follows: (ba) \( \gamma \succeq_A \alpha \) or (bb) \( \gamma \succeq_A \beta \).

(ba) together with (1) directly show that \( \alpha \) is a weakly dominated strategy in the conventional (strategic) sense. So assume instead (bb): \( \gamma \succeq_A \beta \) and \( \gamma \sim_A \delta \sim_A \alpha \). After a deviation to \( c \) payoffs are possibly higher but never lower than they are at \( a \) and, again, \( \alpha \) is a weakly dominated strategy. Hence, \( \alpha \not\in SGE \).

To complete the induction, we note that for \( v_{t-1} = v_{T-1} \) and terminal nodes \( a \) and \( c \) the starting conditions \( Y(a) = SGE(a) \) and \( Y(c) = SGE(c) \) trivially hold.

**Proof of proposition 7** In the text we have shown that \( \alpha \in \Sigma V(\preceq) \) does not imply \( \alpha \in \Sigma V(\succeq) \).

\(^{19}\)That is, neither condition aa) nor ab) of part a) apply.
For a case where $\alpha \in \Sigma V(\preceq^Y)$ does not imply $\alpha \in \Sigma V(\preceq)$, consider example 3: $V$ is multivalued at node $c$ with $V(c, \preceq) = V(c, \preceq^Y) = \{\gamma, \delta\}$. $\alpha$ is excluded in $V(\preceq)$ because it is (optimistically) dominated by $\gamma$ but $\alpha$ is included in $V(\preceq^Y)$ because it is not conservatively dominated by $\gamma$ and $\delta$.

Proof of proposition 8 To show that WCSSP neither includes nor is included by OSSB, we provide two examples:

a) Show that WCSSP does not include OSSB

$\alpha \in WCSSP$ but not in OSSB: Consider example 2 with preference order (a), i.e., $\gamma \succ_s a \succ_s \alpha \succ_s \alpha$. In this case, the unique OSSB is $\sigma^{OSSB} = \{\gamma\}$ but there are two WCSSP: $V^1(\preceq^Y) = \{\alpha, \delta\}$ and $V^2(\preceq^Y) = \{\gamma\}$.

b) Show that OSSB does not include WCSSP

Consider an extension of figure 2 where $a$ has a direct predecessor $v_0$. $S_0$ may decide to move to node $a$ or it may choose $\pi$. The payoffs for $S_0, S_a, S_c$ are $u(v^a) = (4, 2, 1), u(v^\delta) = (3, 1, 1), u(v^\gamma) = (3, 3, 1)$ and $u(v^\pi) = (3, 1, 1)$.

$\pi \in OSSB(v_0)$ but not in $\Sigma V(v_0, \preceq^Y): \alpha \in V(v_0, \preceq^Y)$ because all continuations are weakly preferred to $\pi$ but $\alpha$ is not dominated by $\pi$ via $\preceq^Y$, hence $\pi \not\in V(v_0, \preceq^Y)$. But with $\alpha$ blocked in OSSB (see part ba) of this proof above), $\{\pi, \gamma\} = \sigma^{OSSB}(v_0)$.

Proof of proposition 9 Following Delver and Monsuur, define $Q^0(\preceq\preceq)$ as the set of top elements with regard to $\prec\prec$: $\alpha \in Q^0(\preceq\preceq)$ if $\beta \in Q^0(\preceq\preceq)$ and $\beta \prec\prec \alpha$ implies that $\alpha \prec\prec \beta$. By construction, there is no element in $Q(\prec\prec)$ is dominated via $\prec$ by an element outside of $Q(\preceq\preceq)$. Clearly, there is such $Q(\preceq\preceq)$: either $\prec$ has a maximal element or there is a cycle involving elements at the end of a $\prec$-chain.

In the proof of lemma 1, replace $M_X^t$ with $Q^0(\preceq\preceq)$ and follow the other steps. At each step $t$, construct the top elements $Q^t$ elements in $X^t$ replacing the maximal elements $M_X^t$.

Following the steps of the induction in the proof of proposition 1, shows that $V(\preceq\preceq)$ exists. Note, that at each step, $V(X(v_t), \preceq\preceq)$ is multivalued if the solution includes a cycle.

Proof of proposition 10 Observe that by completeness of order, $\beta \ll \alpha$ if and only if $\beta \ll \alpha$. Moreover, $\ll_{V}$ if and only if $\beta \ll \alpha$ and $\alpha, \beta \in V$. Let $(\ll_{V})^a$ be the asymmetric

\[20\text{In Delver and Monsuur, } Q \text{ is defined as a set without "incoming arcs" where an "incoming arc" stands for the relationship "is dominated by an outside element". If elements in any given cycle are dominated, mark the cycle as "previously crossed" and move to a cycle at a higher level. Because } X \text{ is finite, either there is a highest level cycle or } \prec \text{ connects to a previously crossed cycle in which case all elements between the higher-level and the previously crossed cycle are included in } Q^0(\ll_{V}).\]
Any set $W \subseteq X$ is stable for $(X, (\ll_V)^a)$ if and only if it is stable for the relation $(X, \ll)$. Considering the symmetric relation, by the internal stability condition of socially stable set, $\alpha, \beta \in V(\ll, \ll)$ implies $\beta \ll \alpha$. By completeness of order, $\alpha \ll \beta$ implies $\beta \ll \alpha$ if and only if $\alpha, \beta \in V(\ll)$ and $\alpha \ll \beta$ implies $\beta \ll \alpha$. So assume that $\alpha \leftrightarrow \beta$ and $\alpha, \beta \in V(\ll)$. We have to show, that stable sets exist with $\alpha \in V(\ll)$ and $\beta \in V'(\ll)$.

As $\alpha \not\in V'$, suppose $\delta \in dom(\alpha)$ dominates $\beta$. However, this implies $\beta \ll \delta \ll \alpha$. But because $\alpha \leftrightarrow \beta$, this contradicts acyclicity. Hence, $\beta$ is maximal in $V(\ll) \cup dom(\alpha)$. As this goes for all elements in $V(\ll)$ and $dom(V(\ll)) \cup V(\ll) = X$, $\beta$ is maximal in $X$ and a stable set including $\beta$ can be constructed following the steps in the proof of lemma 1.

**References**


