Coalition Formation Processes in Real Time: Generalizing Subgame Perfection for Dynamic Paths

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Preliminary Version

Abstract

In this paper we develop the stable set defined on dynamic paths as a solution concept for coalition formation processes in real time. Our model accommodates myopia and sophisticated farsightedness. For the case where only a singleton moves at each node, our solution coincides with the set of subgame perfect paths.

Keywords: *Coalition Formation, Farsightedness, Dynamic equilibrium*

JEL codes: *C71, F15.*

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1 Introduction

How should we think about coalition formation processes in real time? Coalition formation processes in an atemporal environment typically assume that agents are only willing to move along such paths along which they improve their well-being.[1] Coalition formation processes in real time such as introduced by Konishi/Ray (2003) - although focusing on absorbing states - make a similar assumption about admissible paths. In an atemporal environment, where agents may travel any path, this assumption is innocuuous if paired with an assumption about players' assessment of other players' subsequent moves: Players are either "optimistic" and potentially ignore possible threats posed by moves of other players or they are "conservative" and think adverse moves by other players are a distinct possibility. Yet when analysing coalition formation processes in real time, the focus on indirectly dominating paths implies that agents who participate in the process are farsighted but unsophisticated when it comes to assessing threats. Thus, when we are interested in the path leading to an equilibrium state, analogous considerations to those taken into account in the non cooperative concept of subgame perfection may also be invoked in a coalition formation process.

In a parallel development, a number of recent papers have shown how reference to an explicit bargaining protocoll can obtain insights into coalition formation processes: Gomes/Jehiel (2005) have recently shown that depending on the bargaining protocol and in the presence of negative externalities between coalitions inefficient states can be stable in Markov perfect equilibrium. The dynamic path may involve enduring a temporary inefficiency even when agents can anticipate the inefficiency and can write a contract which avoids this outcome. Alimbekos/Madumarov/Pech (2015) demonstrate in a model with transferable utility, that an open-loop bargaining protocol in the sense of Baron/Ferejohn (1989) allows agents to internalize the negative externality. More generally, Acemoglu/Egorov/Sonin (2012) show in a dynamic environment with transfers that, while inefficient processes are possible, there exist bargaining protocols with which the dynamic system immediately moves into the dynamically stable state.

If a solution concept for a coalition formation process is to make predictions for an institution-free bargaining environment it should, therefore, allow agents to negotiate away from inefficient situations to the extent that the inefficiency is the result of a credible threat. Yet while subgame perfection offers a convincing solution for non cooperative games which establishes the credibility of threats and accounts of them, there is no equivalent notion for coalition formation processes.

In our leading example a formateur has to find an optimal sequence in which to

[1] Such paths are said to indirectly dominate their starting point, see Chwe (1996), Xue (1998) or Pech (2012, 2015).

form a customs union in an environment where coalitions exert negative externalities. In this example, players may be jointly better off by immediately accepting the absorbing state if this is proposed to them rather than going through a period of punishment when the formateur carries out his threat to unleash external effects. The consequences of such threats are ignored in standard models of dynamic coalition formation.

It turns out that our solution concept − the stable set defined on dynamic paths − is capable of taking account of such strategic considerations in a similar way as subgame perfection does in extensive form games. Our modelling approach allows agents to entertain myopia or (sophisticated) farsightedness and is flexible enough to incorporate different institutional constraints.

Section 2 introduces three motivating examples. Section 3 compares our modelling approach to the literature. Section 4 sets up our dynamic coalition formation model. Section 5 provides first results. Section 6 applies our results. Section 7 discusses the relationship of our solution concept with subgame perfect equilibrium. Section 8 presents a refinement of our solution concept.

2 Applications

2.1 Customs Union Formation

Consider the following customs union formation game inspired by Aghion, Antras and Helpman (2007) which is played over two periods with a status quo point a_0 as depicted in figure 1. If each agent stays singleton − corresponding to the vertices labelled "a" with time indices 0, 1 and 2 − each realizes a value of 1. The total value of the grand coalition is 6 and the total value if coalition {A,B} forms the customs union is 4. Moreover the core customs union {A,B} exerts an externality on C who realizes a pay off of zero. Efficient outcomes are in node c_1 or d_1 where the grand coalition forms in the first period. c_1 can be realized without side payment but d_1 involves a sidepayments between the members of the grand coalition. Possible moves by coalitions are illustrated in figure 1. The inducement correspondences represented by solid arrows represent moves where some coalition is effective. The broken arrows represent the default path: In any node, if no coalition moves, the empty coalition brings about a move along the default path in that node. Pay offs associated with a position (such as v_d in d_t, $t = 1, 2$) are realized each time the position is realized. Agents have a common discount factor $\delta = 1$.

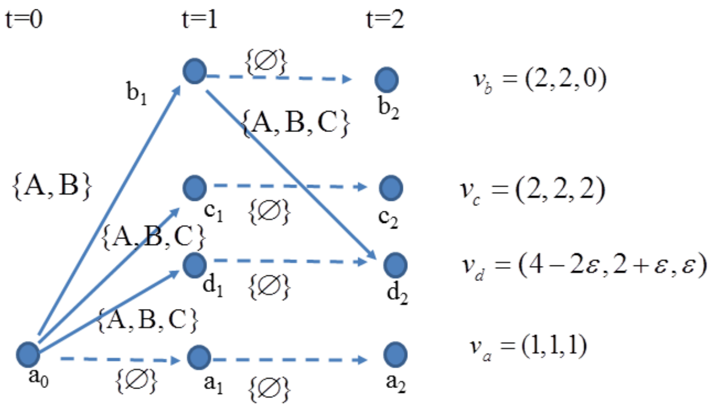

Figure 1: Customs Union Formation Game

A and B prefer the inefficient path $a_0b_1d_2$ to the efficient "equal split" path $a_0c_1c_2$. This is the path which indirectly dominates the status quo point in the sense that the coalition which moves in each point predicts that a subsequent coalition will be willing to move to a node such that the resulting path improves on the default path. The interpretation is that because of the externality, once b_1 is reached, C will be happy to participate in a move to point d_2 in period $t = 2$. This path also corresponds to the solution obtained by Aghion, Antras and Helpman in a framework which assumes a bargaining protocol with A as proposer where proposals are put against the current status quo in $t = 0$ and $t = 1$ and can only be accepted or rejected in a single round of bargaining.[2]

The argument against this solution is that C should realize, that if the path $a_0d_1d_2$ is proposed to her, she is better off than with the path $a_0b_1d_2$. Alimbekos/Madumarov/Pech (2015) argue that if the proposal maker chooses as bargaining protocal an "open-loop" rule and proposes the core customs union as a contingent outcome in the first round of bargaining this will pursuade player C to

[2]This protocol has been labelled "closed-loop" in Alimbekos/Madumarov/Pech (2015).

accept d_1 in the second round of bargaining.[3] It, therefore, makes sense to consider this solution also as a possible outcome in an institution-free bargaining process.

Ultimately, it would seem desirable to have a solution concept for cooperative dynamic games which allows to take into account the same kind of reasoning - here the availability of a credible threat to create a core customs union - which would enter agents' considerations if the game were modelled as an extensive form game with a distinct institutional structure. Whilst different institutions may still support different - and some of them inefficient - solutions, the focus on paths which indirectly dominate the default path rules out the consideration even of credible threats.

2.2 A Public Good Game

The argument about credible threats is crucial for the question of whether or not the system immediately moves from the status quo into the efficient state. A related question is whether the efficient state, once reached, is stable in the sense that agents do not want to move away from this state once it is reached. In the example, it is clear that A and B would want to move neither to state b nor to state c. The presence of externalities may induce agents to move away from an efficient state. This can be demonstrated in a simple public good example which is due to Ray (2007).[4]

Assume that the status quo situation a is associated with players forming a set of singletons $\{\{A\},\{B\},\{C\}\}$ and the outcome vector $(1,1,1)$. If the grand coalition forms, it can provide a total pay-off of 9. In node c, the equal-split allocation is realized. If C leaves the grand coalition or if coalition $\{A,B\}$ forms and C stays as a singleton, the outcome is $(2,2,4)$. This game is depicted in figure 2. Whenever the grand coalition with equal split allocation has formed, C has incentives to move away. In this case, the prediction is that the grand coalition is unstable and that eventually c_2 will be realized.

Assume, however, that the grand coalition can effect another node d either from node a_0 or from node c_1 where its value is split unequally with pay-off vector $(2\frac{1}{3}, 2\frac{1}{3}, 4\frac{1}{3})$. Now, the unequal split allocation for the grand coalition is preferred

[3] Gomez/Jehiel observe that their inefficiency result only obtains if agents cannot sign general spot contracts where allocations and transfers can be made contingent on players' responses.

[4] Gomes/Jehiel (2005) provide a general inefficiency result which applies to Markov-stable states in an open-horizon model, stating that the set of stable (ergonic) states involves inefficiencies if a move away from some efficient state by at least two agents involves negative externalities on agents who are not effective in brining about this move. Their argument does not directly apply to our examples as they focus on equilibrium proposals of a proposal maker process yet the example may be used to check whether in a particular situation an agent has a strategic move which may induce another agent to accept a lower pay-off. Typically, this will be the case if the agent has a move which inflicts a lower pay-off on an outsider.

by everybody to node c_2. In this case we would predict that the grand coalition forms, either with unequal split allocation in the first period or with the unequal split allocation reached after one period in the equal split state.[5]

An interesting aspect of this example can be brought out in a symmetric extension of this game where any player can take on the role of player C for every division of the value of the grand coalition there is some player who wants to deviate.[6] In this case, the negative externality inflicted by the player who moves away from the efficient state and the impossibility of accommodating every player who may raise a threat against an allocation in the grand coalition is the reason why the system does not stay in the efficient state.

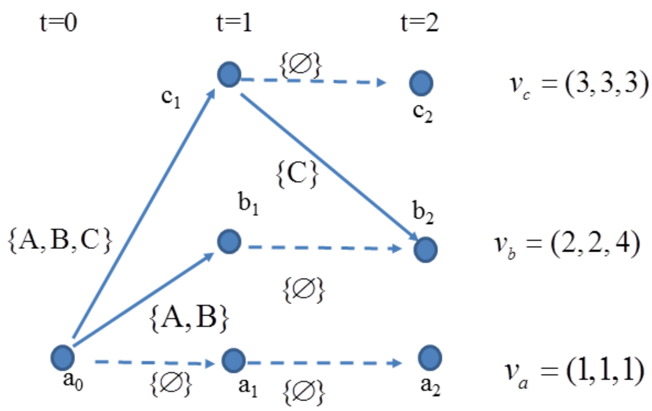

Figure 2: A Public Good Game (Ray, 2007)

2.3 Political Institutions

Institutional design affects the game in figure 1 in the form of rules about which set of agents can move in any node. Farsightedness of the coalition who can move

[5]In this case the paths $a_0c_1b_2$ and $a_0c_1d_2$ and $a_0d_1d_2$ indirectly dominate the default path $a_0a_1a_2$.

[6]This argument affects every division including the devision at node d discussed above.

away from an inefficient state may prevent such a move if the coalition foresees that the coalition who is effective in the efficient state would like to move away from it. The following example is due to Acemoglu/Egorov/Sonin (2012): The status quo (monarchy) gives the elite E and the middle class M a pay-off of 1 each. Both the elite and the middle class prefer constitutional monarchy where each gets a pay-off of 2. Yet in constitutional monarchy, the middle class would opt for full democratization with an associated pay-off for E and M of $(0, 3)$. This game may start in any period and indefinitely stays in the democracy node once it is reached. Clearly, if E is sufficiently patient, it does not want to move away from a because its long-term pay-off from moving to c - taking into account the subsequent movement to b is lower than the pay-off from staying in a.

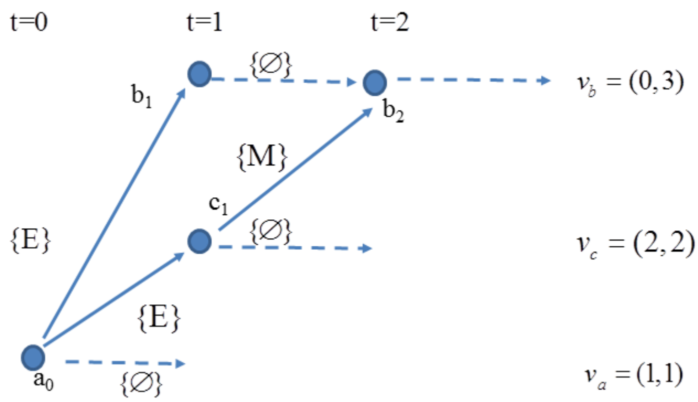

Figure 2: The Democratization Game (Acemoglu/Egorov/Sonin, 2012)

As the examples demonstrate, our concern here is with inefficiencies from ignoring credible threats on a path to a dynamically stable state rather than with strategic moves which keep the economy away from a dynamically stable state.

3 Comparison to the literature

Konishi/Ray (2003) and Gomez/Jehiel (2005) model Markov-perfect strategies in a stationary environment with probabilistic moves and randomized strategies. The present model is more general in that it does allow for non stationary environments and allows for consideration of strategic choices which are not Markov-perfect but it is less general in that it does not allow for mixed strategies.

Greenberg (1989) and Tadelis (1996) analyse optimistic social standards of behavior (OSSB) in tree situations corresponding to simple game trees or repeated games. Greenberg shows that the unique OSSB is a (weak) subset of the set of subgame perfect equilibrium paths (where paths losely correspond to strategies). Tadelis shows that the OSSB for a repeated game is a refinement of subgame perfection with desirable properties such as Pareto-dominance. The framework developed in this paper defines time paths where agents $-$ in each non-trivial node $-$ choose between different continuations of these time paths.

Xue (1998) analyses social standards of behavior in a graph situation. He assumes that in a non trivial node agents can "defect" to a different path. The consequence of this modeling choice in the theory of social situations is that if agents can defect from either of two different paths $-$ a setting where subgame perfection would make a non-unique recommendation $-$ the solution is quiet on which path agents follow, i.e. it assigns the empty set as solution in such a node. The approach followed in this paper allows for the existence of multiple stable sets which more closely corresponds to the idea of subgame perfection.

As we show, our solution generally exists in graphs where at each node only one agent moves and makes a $-$ possibly non-unique $-$ recommendation. A solution fails to exist in the case where the preference relationship on paths is cyclical $-$ for example involving three different coalitions wanting to move from one node along three different paths $-$ as in this case the stable set does not exist.

4 The model

We consider an n-person game of perfect information without chance moves. Let Z be the set of nodes and Z_t be the finite set of nodes which correspond to period t. Π is the set of feasible paths connecting these nodes and $X(v_t)$ is the set of paths which can be reached from node v_t in period t. Agents have preferences $\succsim_i (v_t)$ defined on $X(v_t)$. This includes the case of hyperbolic discounting of payoffs accruing in the nodes which are reached along paths starting in v_t as a special case. Feasibility of paths may reflect institutional or factual relationships between situations over time. For convenience we assume that a path assigns at least one vertex to each time period subsequent to the path's origination. Let a and b denote two nodes. An effectivity relation $a \xrightarrow{S} b$ signifies that in situation a, coalition S can bring about situation b. Moreover, $a \xrightarrow{S} b$ implies that the arc ab is subset of some path and that b belongs to the time-indexed set Z_{t+1} subsequent to a.[7] That is, each possible move of a coalition must belong to some path and nodes which can be induced from one another must be subsequent in time. Moreover, if no coalition moves in a node, a default path is followed as defined in definition 3.

Definition 1. Assume that node b is a direct successor of a on path α. A (truncated) path $\gamma_{|a}$ dominates path α via $\triangleleft$, i.e. $\alpha \triangleleft \gamma_{|a}$, if there is a node $a \in \alpha \cap \gamma$ and $a \xrightarrow{S} c$, $c \in \gamma$, $c \notin \alpha$ and for the truncated paths $\alpha_{|b}$ and $\gamma_{|c}$: $\gamma_{|c} \succsim_S \alpha_{|b}$.

So $\gamma_{|a}$ dominates α if there is a junction a at which path γ diverges from α and there are agents which are capable of and willing to veering off the path.

At any particular node, the domination relation may be cyclical. Cyclicality poses a problem for existence except in the case where there is one path originating in a which dominates all others, so we adjust our definition of cyclicality of $\triangleleft$ accordingly:[8]

Definition 2. $\triangleleft$ is cyclical at node a if there are $K > 2$ paths $\alpha_k \in X(a)$ such that (1) there is a sequence $\alpha_k \triangleleft \alpha_{k+1}$, $k = 1, ..., K-1$, with $\alpha_K \triangleleft \alpha_1$ and (2) there is no path α_j such that $\alpha_j \succsim_{S_j} \alpha_k$ for all $k = 1, ..., K$.

We assume that for any vertice v'_t there is a default arc $v'_t v'_{t+1}$ which is followed if no coalition which is effective at v'_t wants to move. This is a quite natural assumption to make if coalition formation takes place in real time. The default path takes the place of the default position in stationary models of coalition formation where agents want to move away from a position if all agents necessary to effect the move approve of it. This setting is very general and includes the case of repeated games of Tadelis (1996) where after each termination node of the stage game another, equivalent, stage game starts and the stationary case of Konishi/Ray (2003) and Gomez/Jehiel (2005) where agents reach a strategically equivalent position in the next period if they don't move away from a node.

Definition 3. In each node a there is one path which originates and for which the empty coalition is effective. We call this path default path and use $\omega(a)$ as a label for this path.

[7] With a slight adjustment of definition 1 we can also consider situations where "procedural" moves to a node b in the same time-indexed set of situations Z_t as node a are possible.

[8] In the case of an even number of paths in the cycle, we may interpret the inclusion of different paths in the same set as representing contemplated possibilities rather than actual actions (I am grateful to Anne van den Nouweland for this suggestion). Another possibility is to say that either there is a mixed strategy with pay-offs $\Sigma_{k=1}^{K/2} s_{2k-1} U(\alpha_k) > \Sigma_{k=1}^{K/2} s_{2k} U(\alpha_{2k})$, or vice versa, or both mixed strategies are admissible (where the s_k's are chosen to maximize the product of utiltity increments over the alternative).

We assume that the default path dominates another path α in some vertice if one agent necessary to bring about α prefers the default path:

Definition 4. Let α be a path originating in node a with effective coalition S_α and $\omega_{|a}$ be the corresponding default path. If $\alpha \not\succ_{S_\alpha} \omega$ then $\alpha \lhd \omega$.

In the appendix we consider an alternative definition where the default path dominates a path at some vertex if it dominates all paths originating in this vertex. We show that replacing definition 4 with the alternative definition does not affect the solution set when there is no cycle.

The notion of a default path is natural in a dynamic environment. However, for our analysis we can treat the default path as any other path and we could even suspend with the default path altogether without affecting any of the results of the paper.

Like for any other path, we have to exclude the possibility of the default path being included in a cycle:

Example. 1. Consider the case where in node a $\{1,2\}$ can induce α with pay off vector $(2/3, 1/3, 0)$, $\{2,3\}$ can induce β with pay off vector $(0, 2/3, 1/3)$ and the default path is associated with the pay off vector $(1/3, 0, 2/3)$.

Here, β dominates α. The default path is dominated by α and - by definition 4 - dominates β. In this case, applying definition 4 removes β as the obvious candidate for the solution and creates a cyclicality. This is quite justifiable: Agent 3 will be happy to consider rejecting path β for the default path as long as α is not the "predicted" path which prevails. Of course, α is not predicted as long as it is dominated by β.

Next, consider the case where no coalition which is effective for a path starting in some node, actually wants to move along the path for which it is effective. In this case, the default path may dominate the other paths via $\lhd$ but this is neither guaranteed nor is it crucial for our results:

Example. 2. Let the inducement correspondences for path α be a_0 be $a_0 \xrightarrow{S} c$ and for path β be $a_0 \xrightarrow{T} b$. Moreover, let the preferences be $\beta \succ_S \alpha$ and $\alpha \succ_T \beta$. In this case, no coalition wants to travel along the path for which it is effective.

It is easy to see, that either the default path dominates α (β) or we have $\beta \succ_S \alpha \succ_S \omega$ ($\alpha \succ_T \beta \succ_T \omega$). If the relationship holds for α and β, neither do we have $\beta \lhd \alpha$ nor vice versa, i.e. α and β are in the set of paths which are undominated via $\lhd$. If the relationship is violated for T (say, $\omega \succ_T \beta$) and because T is effective for ω versus β, we have $\alpha \lhd \beta$ (via a move by S), $\beta \lhd \omega$ (via a move by T) and $\omega \lhd \alpha$ (via a move by S). Hence, we have a decision cycle.

Definition 5. A set $V \subset \Pi$ is stable if it is internally stable and externally stable. It is internally stable if $\alpha \in V$ implies that there is no $\beta \in V$ such that $\alpha \triangleleft \beta$. And it is externally stable if for all $\gamma \notin V$ there is $\alpha \in V$ such that $\gamma \triangleleft \alpha$.

5 Results

We present our existence result for the case where at each node possibly different coalitions move. The case of where only single agents move at each node single agents move is a special case where preferences on paths are transitive.

Proposition 1. *Assume that at no node the dominance relation* $\triangleleft$ *is cyclical. Then a stable set for* $(\Pi, \triangleleft)$ *exists.*

Proof. Assume the status quo is a_0, α is a path originating in a_0. By our assumption, there are no (strict) cycles.[9] Hence, there is at least one path originating in a_0 which is strictly undominated with respect to $\triangleleft$ (see example 2). Assume that this path is α and let β be some other path originating in a_0.

By external stability, there is a stable set V such that

(1) $\alpha \in V$ or

(2) there is a node $c \in \alpha$ and a path γ such that $\gamma_{|c} \in V$[10] and $\alpha \triangleleft \gamma_{|c}$.

If (1) is true, we are done. It remains to show that if (1) does not hold, (2) must hold.

Suppose that (1) and (2) fail to hold. If (2) does not hold, by external stability, $\alpha \in V$ for some V, contracting that (1) does not hold.

So consider the (non truncated) path γ in a_0 . By external stability, there either is $\beta \in \Pi(a_0)$ with $\beta \in V$ and $\gamma \triangleleft \beta$ or $\beta \triangleleft \gamma$ with $\gamma \in V$.

□

Definition 1 establishing the dominance relation ensures that if a path is dominated by another path agents could travel this other path. If they are unwilling to do so, this other path is dominated. So the proposition holds for paths of infinite length $-$ as long as the length is countable and we can determine a continuation path at each point at which a dominating path branches off.

Note that there are no other conditions which we need to impose such as boundedness of pay offs: Assume that the path α and ω are both associated with a pay off of ∞ for all players. If we agree assume that agents are indifferent between all paths which yield a pay-off of ∞, each path weakly dominates the other and each path is selected into one stable set. Note that in this case existence is ensured but all paths which indefinitely yield non zero profits may be selected.[11]

[9] Weak cycles such as $\alpha \xrightarrow{R} \beta$, $\beta \xrightarrow{S} \omega$ and $\omega \xrightarrow{T} \alpha$ with $\omega \sim_T \alpha$ do not contradict $\alpha \in V$.

[10] γ might be the default path in c.

[11] Unless another criterion for domination between paths is selected.

6 Application to Our Motivating Examples

In our customs union formation example, path $\{a_0d_1d_2\}$ with a total pay off or $(8-4\epsilon, 4+2\epsilon, 2\epsilon)$ and $\{a_0c_1c_2\}$ with a total pay-off of $(4,4,4)$ do not dominate each other. $\{a_0d_1d_2\}$ dominates $\{a_0^b, b_1, b_2\}$ and $\{a_0c_1c_2\}$ dominates the default path. Hence, there is a unique stable set $V = \{\{a_0d_1d_2\}, \{a_0c_1c_2\}\}$.

Now consider the case where agents are non-sophisticated farsighted and only paths are considered which indirectly dominate the status quo path. Here the set of admissible paths is reduced[12] to $\widehat{\Pi} = \{\{a_0b_1d_2\}, \{a_0c_1c_2\}, \{a_0b_1b_2\}, \{a_0a_1a_2\}\}$. Because $\{a_0b_1d_2\}$ and $\{a_0c_1c_2\}$ do not dominate each other, we have $V = \{\{a_0b_1d_2\}, \{a_0c_1c_2\}\}$.

7 Relationship with SGPE

We can interpret paths in our game as sequences of actions - represented by edges - in an extensive form game.

Proposition 2. *Assume in each node only one agent moves and all nodes other than the terminal nodes give a pay off of zero. Moreover, assume that paths are finite. If all agents are sophisticated farsighted, the sets of stable sets on* Π *(weakly) includes the set of subgame perfect strategies of the corresponding extensive form game.*

Proof. Assume the set of paths $\{\alpha_s, \beta_{s+t}, ...\}$ constitutes a subgame perfect equilibrium (SGPE) and also assume $\gamma_s \notin SGPE$, $\delta_{s+t} \notin SGPE$. Let $SGPE(v)$ be the set of paths starting in node v which are part of this particular subgame perfect equilibrium.

By assumption, $\alpha_s \in SGPE(v_s)$.
(1) Consider an alternative path $\gamma_s \in X(v_s)$. If γ_s is consistent with $SGPE$ for all stages $s+t > s$, i.e. $\gamma_{s+t} \in SGPE'(v_{s+t})$ for some subgame perfect equilibrium SGPE', the decision maker must weakly prefer α. By definition 1, $\gamma_s \triangleleft \alpha_s$. Because all paths which are not part of $SGPE$ are dominated (see next step) and α dominates γ, we can choose $\alpha \in V$, $\gamma \notin V$.
(2) Now consider some path $\delta_{s+t} \notin SGPE, t \geq 1$. It follows that $\delta_{s+t} \not\subset \alpha_s$. Because SGPE specifies a strategy for the contingency that node v_{t+s} is reached there must be a node $v_k \in \delta_{s+t}, k \geq (s+t)$ and a decision maker in v_k such that she weakly prefers some path $\beta_k \in X(v_k)$, $\beta_k \in SGPE(v_k)$. Then it must also be the

[12] The method of constraining the set of admissible paths to indirectly dominating paths starting in a_0 and applying the definition of stability to this set of paths only gives meaningful results if there is no ambiguity of how coalitions move at subsequent stages.

case that $\delta_{s+t} \lhd \beta_k$ and, following step (1), we can choose $\beta_k \in V$. Moreover, because there is at least one step between v_s and v_k and $\alpha \not\lhd \beta$, hence V is internally stable.

Applying the same argument for all paths in $SGPE$ and all paths excluded from $SGPE$ we can show that $\{\alpha_s, \beta_{s+t}, ...\} \in SGPE$ implies $\{\alpha_s, \beta_{s+t}, ...\} \in V$ and $\gamma_s \notin SGPE$ implies $\gamma_s \notin V$. Moreover, because following (2) all paths which are excluded from SGPE are dominated via $\lhd$ by a path which is included in V, V is externally stable. Hence, V is a stable set.

$\square$

For strict preferences on outcomes, our result coincides with the unique backwards induction equilibrium of the theorem of Kuhn and Zermelo. If agents are indifferent between outcomes, mulitple SGPE's obtain and the stable set is not unique.

To illustrate, consider the game tree in figure 3. Path γ connects v_0, v_1^1, v_2^1, path α connects v_0,v_1^2 and v_2^3 and β connects v_0,v_1^1 and v_2^2 and branches off from path α in v_1^2. We denominate the branch $\underline{\beta}$.

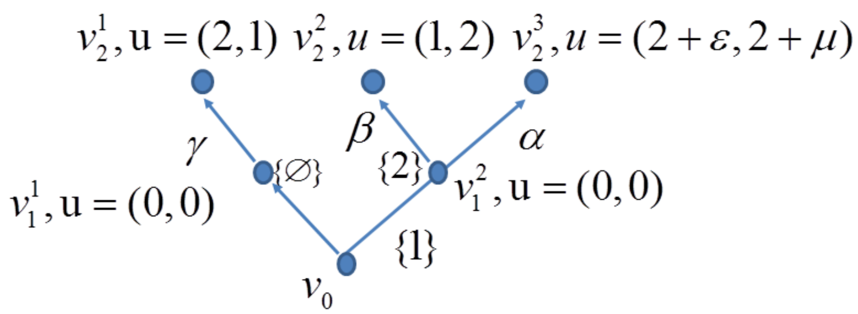

Figure 3: Extensive Form Game

For $\epsilon > 0$, $\mu > 0$ there is one subgame perfect strategy (α). Because α dominates everything else via $\triangleleft$ and is not dominated, $\alpha \in V$ is the only stable path.

For $\epsilon > 0$, $\mu = 0$, there are two subgame perfect strategies: The first consists of γ and $\underline{\beta}$ and the second of α.

β dominates α via $\triangleleft$ and the truncation of α starting in v_1^2, $\underline{\alpha}$, dominates β. Because α dominates γ and $\underline{\alpha}$ also dominates β we have a stable set: $V_1 = \{\alpha\}$. Because $\underline{\beta}$ dominates α, we can select it into V and obtain a second stable set $V_2 = \{\gamma, \underline{\beta}\}$.

For $\epsilon = 0$, $\mu = 0$, γ weakly dominates α in the conventional sense in the normal form of the game and, hence, only γ is selected as an SGPE. Following definition 1, α continues to dominate γ. Hence we still have two stable sets $V_1 = \{\alpha\}$ and $V_2 = \{\underline{\beta}, \gamma\}$.

So the set of stable sets includes paths which correspond to strategies which are not subgame perfect because they are dominated in the conventional sense.

In this example we have not referred to default outcomes. In order to see, how a default outcome affects the results, assume that the status quo is associated with a pay off (1, 1) and that the discount factor is $\delta = \frac{1}{\sqrt{2}}$.

We dynamically extend the game in the following way: If agents do not move away from any node, they indefinitely realize they pay off associated with that node. For example, once they have reached v_2^1, agent 1 realizes a periodic pay off of two indefinitely and agent 2 realizes a pay off of one. So the vector of continuation values associated with v_2^1 is $(2\sqrt{2}, \sqrt{2})$ and the discounted value of following path γ in period 0 is $(\sqrt{2}, \frac{1}{\sqrt{2}})$. This compares to a discounted value of following the default path $\omega = \{v_0, v_1^{def}, v_2^{def}, ...\}$ of $(\sqrt{2}, \sqrt{2})$.

Hence, in the case $\epsilon = 2$, $\mu = 0$ stable sets[13] are $V_1 = \{\gamma, \underline{\beta}\}$, $V_2 = \{\alpha\}$, and $V_3 = \{\omega, \underline{\beta}\}$. There are also stable sets where initially the default path is followed and one of the other stable paths is followed later on.

8 Refining the Stable Set

The following set of definitions closely matches the definition of stable standard of behaviour but avoids assigning the empty set.

Definition 6. A path $\alpha \in \Xi(a)$ if in position a it is undominated via $\triangleleft\triangleleft$. A path α starting position in a with direct successor b is dominated by path γ, i.e. $\alpha \triangleleft\triangleleft \gamma$,

[13] To save notation we ignore truncated paths in the remainder of the game. To the same effect we could restrict attention to non-discriminating solutions in the sense of Tadelis (1996), Greenberg (1989), i.e. if a position repeats itself over time, then the solution admists the same recommended outcome paths for that position.

if $a \in \alpha \cap \gamma$ and $a \xrightarrow{S} c$, $c \in \gamma$, $c \notin \alpha$ and for all paths $\delta_k \in \Xi(c)$: $\delta_k \succsim_S \beta_i$ $\forall \beta_i \in \Xi(b)$ with at least one strict inequality.

If all paths starting in node c have a length of one, $X(c)$ and $\Xi(c)$ coincide, thus closing the recursion.

Definition 7. A set $W \in \Pi$ is weakly stable if it is internally weakly stable and externally weakly stable. It is internally weakly stable if $\alpha \in W$ implies that there is no $\beta \in W$ such that $\alpha \triangleleft\!\triangleleft \beta$. And it is externally weakly stable if for all $\gamma \notin W$ there is $\alpha \in W$ such that $\gamma \triangleleft\!\triangleleft \alpha$.

Definition 8. A refined stable set $\widehat{V} = W \cap V$.

Consider the example in figure 5 again:

For $\epsilon > 0$ and with this definition, neither does α dominate γ nor does γ dominate α. So we have $\Xi(v_1^2) = \{\underline{\alpha}, \underline{\beta}\}$, $\Xi(v_0) = \{\gamma, \alpha\}$ and, hence, $W = \{\underline{\beta}, \underline{\alpha}, \alpha, \gamma\}$. The refined stable sets are $\widehat{V}_1 = W \cap V_1 = \{\alpha\}$ and $\widehat{V}_2 = W \cap V_2 = \{\underline{\beta}, \gamma\}$.

For $\epsilon = 0$, γ dominates α and, hence, $W = \{\gamma\}$. The intersection of V_1 and W is empty, therefore $\widehat{V} = W \cap V_2 = \{\gamma\}$.

So in both cases the recommended behaviour in any node v (i.e. $\Xi(v)$) is the same as the recommendation given by subgame perfection. Overall, W collects all (possibly truncated) paths which may be played in any subgame perfect equilibrium.

Consider the following example taken from Tadelis (1996):

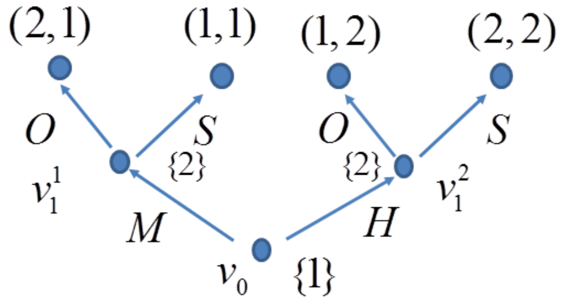

Figure 6: Example

The subgame perfect strategies are $\{S|H, S|M, H\}, \{O|M, S|H, M or H\}, \{O|H, O|M, M\}, \{S|M, O|H, M or H\}$. Note that $S|M$ and $S|H$ together rule out M in a subgame perfect equilibrium in the same way as $O|M$ and $O|H$ together rule out H. Combining the equilibrium strategies to yield entire paths we can write these strategies as $\{SH, S|M\}, \{O|M, SH\}, \{S|H, MO\}, \{O|H, OM\}, \{S|M, OH\}, \{O|H, MS\}$.

It is easy to check that $MS \lhd HS$ and $HO \lhd MO$. This rules out as stable sets $\{MS, S|H\}$ and $\{HO, O|M\}$. Hence, stable sets are $V_1 = \{MO, O|H\}$,$V_2 = \{MO, S|H\}$,$V_3 = \{MS, O|H\}$, $V_4 = \{HS, O|M\}$,$V_5 = \{HS, S|M\}$,$V_6 = \{HO, S|M\}$.

On the other hand, using definition 9 we get $\Xi(v_1^1) = \{O|M, S|M\}$ and $\Xi(v_1^2) = \{O|H, S|H\}$. Thus, neither does the M-path dominate the H-path nor vice versa. So we find $W = \{O|M, S|M, H|M, O|M, MO, MS, HO, HS\}$.

This holds more generally:

Lemma 1. *Assume that the game tree is finite, information complete and at each node only one agent moves. Then at each node a path is in W if and only if it is played as part of a subgame perfect equilibrium. Moreover, a path is stable if and only if it coincides with the equilibrium path of subgame perfect equilibrium.*

Proof. First we prove the "if" part:

Assume at node, v^k_{T-1}, a path/edge γ^k_{T-1} is part of a SGPE. Then γ^k_{T-1} is not dominated via $\lhd\lhd$. Hence, $\gamma^k_{T-1} \in \Xi(v^k_{T-1})$.

By induction, if at a predecessor node $T-s$, path/edge γ^k_{T-s} is part of a SGPE, then $\beta^k_{T-s} = \beta^{T-2,k}_{T-s} \cup \gamma^k_{T-1}$ is not dominated via $\lhd\lhd$ and $\beta^k_{T-s} \in \Xi(v^k_{T-s})$

Assume that a path β^i_{T-s} is also part of an SGPE. In this case, it is undominated via $\lhd\lhd$ and $\beta^i_{T-s} \in \Xi(v_{T-s})$.

Remark: So in v^k_{T-s}, the agent may choose β^i or β^k and neither strategy choice weakly dominates the other.

Now consider any preceding node (such as v_1) and $\alpha^k_1 = \alpha_1^{T-s-1,k} \cup \beta^k_{T-s}$, $\alpha^i_1 = \alpha_1^{T-s-1,k} \cup \beta^i_{T-s}$ and α^j_1.

Now if α^j is unique in SPGE, then it dominates also β^k and β^i via $\lhd\lhd$ and it is uniquely stable.

If α^j is not unique in SPGE (but single valued) and β^k is part of an SPGE, it must be strictly preferred by the decision maker in v_1 to α^j in the case where β^k is realized in a subgame perfect equilibrium. But then α^j does not dominate $\Xi(v^k_{T-s})$ via $\lhd\lhd$. Moreover, if α^j is not unique in SPGE there must be some path $\beta^i \in \Xi(v^k_{T-s}$ which is not preferred to α^j. Hence, α^j is not dominated via $\lhd\lhd$.

Finally, if α^j is not unique in SPGE and also not singlevalued, then there must be a part β^j starting, say, in v^j_{T-s}, i.e. $\beta^j \in \Xi(v^j_{T-s})$ which beats at least one of the paths in $\Xi(v^j_{T-s})$. So neither set dominates the other via $\lhd\lhd$.

Remark: In this last case, there are multiple stable sets such that either α^j or β^j are selected into the same stable set.

Next, we show that the "only if" part holds:

Assume a path $\alpha \in \Xi(v_1)$. So at some successor node v_2, let there be a fork and successors v^1_3, v^2_3 where $\underline{\alpha} \in \Xi(v^1_3)$. Obviously, $\alpha \in \Xi(v_1)$ if not all elements in $\Xi(v^1_3)$ are dominated by $\Xi(v^2_3)$ and in particular $\underline{\alpha}$ is not dominated and, hence, $\underline{\alpha}_{-1}$ is not dominated at v_2.

Clearly, if $\underline{\alpha}$ is not dominated at v_2, it is part of some SGPE.

□

The difference between the stricly stable set W and SGPE is that with the former no continuation is ruled out along a path where subsequent nodes are assigned sets Ξ which are not singular.

If we are interested in obtaining the recommendations of subgame perfection, can use the fact that W collects all paths which are compatible with some SGPE and the fact that the set of stable sets weakly includes stable sets V such that each $SGPE$ corresponds to some set V.

Theorem 1. *Assume that each edge of a game tree corresponds to a section of a path. Moreover, assume that the game tree is finite and that at each node only one* *agent moves and that each move is observed by all agents. Consider the (possibly non unique) stable set of* Π. *Each set* $V \cap W$ *corresponds to one SGPE of a game on the graph* Π.

Proof. Follows directly from proposition 2 and the lemma. □

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9 Appendix

9.1 The difference between the definition of domination in Xue (1998) and this paper

The notion of domination between paths of definition 1 is different from the standard notion of domination such as in Xue (1998) where a path ξ dominates α if there is a coalition S_γ which is capable of and willing to move from node a on path α to a node on path ξ. Figure 6 illustrates the difference between the notions: The section $\xi = ct(\gamma)$ of path γ is dominated by no other path because no path diverges and the same is true of the section $\psi = bt(\alpha)$ of path α. So assume that S_γ prefers path γ to path α and S_α has the opposite preference. If we use definition 1, path γ dominates α and α also dominates γ. If, on the other hand, we use the standard definition, agents can "defect" from α to undominated ξ and, similarly, agents can "defect" from γ to undominated ψ.

Whilst definition 1 allows for two distinct stable sets (one consisting of α and ξ and the other of γ and ψ), the standard definition only allows for a stable set consisting of the truncated paths ψ and ξ both of which dominate via defections of S_γ and S_α the non truncated paths α and γ and also the "default" or "status quo" point a. In this case, the stable set does not make any recommendations of what agents ought to do in $t = 0$ or, if interpreted within the concept of stable standard of behavior, the solution assigned to node a is the empty set.

The closeness of our solution concept defined for the domination relationship of definition 1 to the concept of subgame perfectness is owed to the fact that at any non trivial node agents "choose" between paths rather than "defect" from paths.

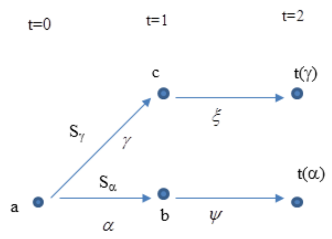

Figure 6: Illustration of the Difference between Definition 1 and the Standard Definition of Domination between Paths

9.2 An alternative notion of domination of the default path

Assume that in node a there are K different paths $\alpha_k \neq \omega(a)$ with effective coalitions S_k. From applying definition 1 and 3, any coalition is effective in bringing about the default path ω which includes at least one agent of the effective coalitions in a, $S_k, k = 1, ..., K$, i.e. the default outcome occurs if no coalition which could move away from a elects to do so. From this, it follows that if $\alpha_k \succ_{S_k} \omega$ fails to simultaneously hold for all $k = 1, ...K$, $\alpha_k \lhd \omega$ simultaneously holds for all $k = 1, ..., K$. Focusing on all feasible paths for deciding on domination is unnecessarily restrictive because some feasible paths will never be taken. Therefore, we focus on paths which are included in the solution.

Now consider replacing definition 4 with the following definition 10:

Definition 9. Let α_k be a path originating in node a with effective coalition S_{α_k} and $\omega(a)$ be the corresponding default path. If for all paths α_j which are included in the solution set V, it is true that $\alpha_j \not\succ_{S_{\alpha_j}} \omega$ then $\alpha_k \lhd \omega$.

Proposition 3. *Assume that $\triangleleft$ is not cyclical at any node. Replacing definition 4 with definition 10 does not affect the solution set.*

Proof. Assume that apart from the default path ω two separate paths α and β originate in node a with effective coalitions S_α and S_β.

Case 1: $\omega \succsim_{S_\alpha} \alpha$ and $\omega \succsim_{S_\beta} \beta$.

In this case $\alpha \triangleleft \omega$ and $\beta \triangleleft \omega$ and $\omega \in V$ unless ω is blocked by some other path.

Case 2: $\omega \succsim_{S_\alpha} \alpha$ and $\beta \succsim_{S_\beta} \omega$.

Now there is a stable set where $\beta \in V$ unless β is blocked. We say β is blocked if there is some other path γ which diverges from β. So that β is blocked implies that there is some other path $\gamma^* = \underline{\beta} \cap \gamma$.

There are two subcases: $\gamma^* \succsim_{S_\beta} \omega$: In this case, if $\gamma^* \in V$, it replaces β. If $\gamma^* \notin V$, it must be dominated by some other path and we repeat the same step. If there is an infinite sequence of domination (in time), we consider the dominating path η_∞ and it must be that $\eta_\infty \in V$ and $\omega \notin V$.

Applying definition 4 leads to the same result: ω now dominates β but it does not dominate γ^* (or the path in V replacing it).

$\gamma^* \not\succ_{S_\beta} \omega$: In this case, we move to case 1 with γ^* replacing β and $\omega \in V$.

Also here, applying definition 4 leads to the same result because ω will be selected as stable element.

□