A Median Voter Theorem for Proportional Representation Systems

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Abstract

We characterize solutions of a political coalition formation game with citizen candidates under proportional representation. Candidates are farsighted and policy-motivated and they decide over running for parliament and forming coalitions. If coalition government is associated with policy uncertainty and candidates dislike uncertainty, political competition results in a solution where the candidate preferred by the median voter runs as a singleton and wins the election. We provide general conditions under which this solution is the unique farsighted stable set of the coalition formation game and obtain an equivalence result for a multi-constituency first-past-the-post system.

Keywords: *Median Voter, Proportional Representation, Endogenous Political Parties, Farsighted Stable Set*

JEL codes: *C72, D72.*

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1 Introduction

In this paper, we show that if policy-motivated citizen candidates dislike the political uncertainty associated with coalition government, political competition in a proportional representation (PR) system works towards a median voter outcome: in the farsighted stable set of a coalition formation game, the candidate whose policy position is preferred by the median voter runs for election and wins as a single candidate. This median voter result for PR systems contrasts with the literature, where median voter results are associated with first-past-the-post - or plurality - voting systems. Yet it goes some way in accounting for the observation that policy outcomes in proportional representation systems tend to be systematically closer to the median voter's preferences than policy outcomes in majoritarian systems.[1]

Downs (1957) formulated his median voter theorem for a two party system where politicians freely select a platform in one-dimensional policy-space to compete for political office under plurality rule. This original result has been extended to cover fairly general conditions, including the case of policy motivated candidates (Calvert, 1985). As voters try to minimize "wasted votes", political competition under plurality voting favors a two party system, a prediction known as Duverger's law (Duverger, 1954). This closes the theoretical argument connecting the institution of plurality voting and the median voter outcome. PR systems, on the other hand, are typically associated with multiparty competition with some parties taking more extreme positions than their voters (Kedar, 2005) and the theoretical account predicts the median voter outcome - or a tendency towards the median voter's position - only as one of a multitude of possible outcomes.[2]

One possible reason for the discrepancy between theoretical prediction and empirical results is that the median voter argument for plurality voting systems overstates the centripetal forces under this institution. The prediction of two-party competition does not generally correspond to observation[3] and relaxation of the assumptions of the Downsian analysis reveals centrifugal forces: divergence may be the result of voter abstention (Laussel, Le Breton and Xefteris, 2017), internal party processes such as primaries (Adams, Merrill III and Grofman, 2005) or uncertainty about the position of the median voter. The latter leads to diverging policy positions in the case where politicians are policy motivated (Roemer, 1997).

At the same time, there are centripetal forces in PR systems. Checks and bal-

[1]See Huber and Powell (1994) and Powell and Vanberg (2000).

[2]See Bandyopadhyay, Chatterjee and Sjöström (2011) or Baron (2018).

[3]There are smaller parties running for election in the UK and India, both of which use plurality voting, and even in US presidential elections, third party candidates run quite often.

ances within a multiparty government work against the adoption of extreme policy proposals (Tsebelis, 1995, Huber and Powell, 1994). Policy motivated representatives can strategically exploit this property in a government formation process in an elected assembly (Pech, 2012). In this paper we show that electoral forces in PR systems also work in favor of the median voter position, if we assume sophistication and policy orientation on the side of candidates.

As a departure from the modelling assumptions of Downs (1957) and Calvert (1985), we establish our median voter theorem in a variant of the citizen candidate model introduced by Besley and Coate (1997). Candidates do not strategically select their policy platform but, rather, a credibility constraint applies and a candidate implements their preferred policy when given the opportunity after the election. Instead, the decision for candidates is whether to run for election and which coalitions to form. In case a coalition government emerges, it is associated with a lottery based on the equilibrium proposals that coalition members make in post-electoral bargaining. Candidates dislike uncertainty over policy and prefer a distribution of possible policy outcomes which is less dispersed and closer to their ideal point. Under fairly general conditions on voter behavior and the policy selection process, we show that potential candidates want to compromise by exiting the race up to the point where the candidate preferred by the median voter wins the election as a singleton.

While our model does not predict the empirical variety of parties competing in PR systems, it explains the observed tendency of such systems towards moderation. This contrasts to the related model of Hamlin and Hjortlund (2000) which identifies conditions where PR results in multiple candidates running for election but fails to account of the moderating effect of PR systems. As a theoretical contribution, our paper isolates the effect of policy motivation of political agents, assuming that they act under a credibility constraint. For comparison, we develop a simple multi-district model with plurality rule and find convergence of results for both models. Ultimately, our median voter result provides a reference point where, as in the case of the Downsian median voter result, variations of the underlying assumptions result in predictions which account of the differences in observed electoral outcomes. We discuss some variations of our modelling assumptions in section 4.

We impose some technical assumptions to derive our results: voters are ordered by their ideal points which are continuously distributed on the unit line. A finite subset of voters may serve as candidates.[4] Potential candidates are farsighted when they decide whether to run for election or to form coalitions at the pre-election stage,

[4] Our approch combines aspects of Hansen (2022) where the set of potential candidates is finite but the position of the median voter's ideal point is perceived as continuously distributed, and Hamlin and Hjortund (2000) where citizens' ideal points are continuously distributed.

that is they predict the consequences of their decisions on the actions of other potential candidates. The cost of running for a candidate is either small or zero. We assume that there is a correspondence which uniquely maps a coalition structure into a lottery of policy outcomes. We motivate this assumption by introducing two alternative policy selection scenarios: in our first scenario, candidates may agree to form firm pre-electoral coalitions and, after the election, members of the winning coalition are selected to propose a policy with a probability corresponding to their relative vote share. In the second scenario, a policy is negotiated in full parliament after the election. In either case voters cast their vote for the candidate whose (equilibrium) proposal they prefer but we also present a median voter result for the case where voting strategies form a strong Nash equilibrium. While our basic result is driven by the inability of coalitions of multiple candidates to commit to one policy point and the ensuing policy uncertainty, we also derive a median voter result for the case where pre-electoral coalitions can commit to a policy point in the Pareto-set of the coalition members. Because farsighted stable set is based on the assumption that candidates, when contemplating a deviation, hold optimistic expectations about subsequent deviations by other candidates we also explore the implications of conservative expectations: candidates, only deviate if all possible subsequent deviations to a stable outcome are better than the point of departure.[5] While the median voter solution is also stable for conservative behavior,[6] other coalition structures may be stable as well.

That prospects with a coalition government are best modeled as a lottery appears natural because after an election parties in a coalition government have to compromise on their announced party platform. While coalition treaties could, in principle, provide a tool to commit to a joint policy platform, they are often not negotiated before an election. In practice, post-electoral negotiations in coalitions typically involve the assignment of cabinet portfolios between the coalition parties and ministers enjoy some autonomy in initiating government policies.[7]

Previous contributions to the literature have mainly applied Nash equilibrium to model political equilibrium in proportional representation systems.[8] Farsighted

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[5]Optimism or conservatism are unrelated to risk aversion with regard to lotteries, see, e.g. Greenberg (1989), chpt 11.

[6]Our model of conservatism is largest consistent set, see Chwe (1994).

[7]See Laver et al (2011). Laver and Shepsle (1994) write that "whether legislators consciously perform (...) calculations or not, they must form beliefs about cabinet decision making in different potential executives so that they can forecast what each will do if given the keys to government buildings".

[8]For example, Bandyopadhyay, Chatterjee and Sjöström (2011) consider post-electoral bargaining for the cases of pre-electoral coalitions and post-electoral bargaining and compare policies under

stable set makes stronger assumptions than Nash equilibrium about the ability of agents to coordinate their actions and to plan ahead. The relevant test is stability of a coalition structure against alternative structures rather than proofness against myopic individual deviations. Yet political coalition formation processes will typically involve a degree of strategic thinking. Much of our analysis relies on the willingness of potential candidates to compromise in order to reduce the spread of potential coalition governments in policy space, which we model as a decision of potential candidates to exit the race. Voters in proportional representation system have shown some willingness to switch their support to a center party when the probability that an ideologically close center-left or center-right party is included in a coalition government increases - something which can be described as a Duvergerian effect in PR systems (Bargdsted and Kedar, 2009). In our model, we assume that citizen candidates are willing to make similar strategic compromises. That candidates exit the race, for example by uniting behind one candidate in order to minimize lost votes in an electoral district is a phenomenon typically associated with first-past-the-post systems.[9] But also in PR systems, parties may form pre-electoral coalitions by running on a joint party list.[10] Overall, pre-electoral coalitions may take different forms that narrow down the spread of policy lotteries associated with post-electoral negotiations to a different degree.[11] Parties may even strategically vacate policy positions for strategic reasons such as preparing for post-electoral coalition negotiations.[12] Finally, although this paper does not provide a model of the formation of parties as coalitions of distinct candidates, we may motivate the idea of a candidate exiting the race as a decision to unite behind a party figure head who ends up representing the political position of the party.[13]

PR and majority voting. They predict the median party to form the government if there is sufficient ideological divergence between left and right. Other contributions comprise Austen-Smith and Banks (1988), Baron and Diermeier (2001), Baron, Diermeier and Fong (2012) and Cho (2014a).

[9]Examples where parties or candidates in first-past the post elections form (possibly implicit) pre-electoral coalitions include the UK parliamentary elections of 2019 where the Brexit Party did not contest Conservative-held constituencies but also France, Mexico and South Korea (see Shin, 2019). For other examples see also Bormann and Golder (2013).

[10]Golder (2004) cites the UDF and PRP in France and Bandyopadhyay (2011) Greece, Portugal and, to a lesser extent, the Netherlands, as examples.

[11]See Golder (2006). Strom and Müller (2000) find that one third of coalition governments in their sample have arranged some form of pre-electoral coalition agreements.

[12]For example, the German Christian Democratic party changed its position on the timeline for replacing nuclear engergy in 2011 in a move which was seen as appealing to the electorate as well as facilitating future negotiations with the Green Party over coalition government.

[13]In Shin's (2019) model of pre-electoral coalition formation in presidential elections candidates may withdraw strategically to throw their support behind another candidate, although they may

Section 2 introduces our model. Section 3 states the main results. Section 4 considers alternative assumptions. Section 5 concludes.

2 The model

The policy space is $X = [0, 1]$ with policy realizations $x \in X$. Voter and candidates are identified by their ideal point $\widehat{x}$ on X where we assume that voters' ideal points are continuously distributed with density $h(\widehat{x})$ and median m. We define preferences in policy space as $U(x; \widehat{x}) = -(x - \widehat{x})^2 - c$, where $c \geqslant 0$ is a cost incurred by a citizen-candidate who runs for election.[14] Among all citizens there is a finite set of potential candidates, N, with cardinality n. We refer to the ideal point of candidate i as $\widehat{x}_i$,[15] and assume that there is a strict ranking of candidates along the unit line. We denominate m_P the candidate in N whose ideal point $\widehat{m}_P$, among all candidates, is preferred by the median voter. Moreover, we assume that this preference of the median voter is strict.

A coalition structure is represented as a partition π of N which results from the decision of potential candidates to run or exit the race jointly or individually or to form coalitions between them. Our interpretation is that a potential candidate who runs is identified as a political party. A typical partition has the form $\pi = \{S_1, .., S_J, D\}$ where S_j are (pre-electoral) coalitions, which are possibly singletons, and D is the set of potential candidates who do not run. Π is the set of all partitions of N. In the remainder of the paper we drop D to simplify the notation of partitions.

We analyzse our model in three stages: at stage 1, potential candidates decide on whether or not to run and whether to form (pre-electoral) coalitions, resulting in a partition π. In stage 2, citizens vote given π. At the final stage, elected candidates choose a policy. We analyze the three stages in reverse order:

2.1 Policy Choice

Each partition π is associated with a lottery $\ell = \{(x_i, p'_i)\}_{i \in S'}$ of policy outcomes where a typical element is described as a policy proposal x_i by candidate i who is included in the set of potential proposal makers S' and weight p'_i which corresponds to the probability that candidate i is selected to make a proposal. We consider two scenarios with different selection protocols but in section 2.3 we provide a generalization which includes the two scenarios as special cases.

compromise on policy.

[14] We assume that a candidate incurs a cost only if they actually stand for election.

[15] For convenience we write $U_i(x)$ for $U(x; \widehat{x}_i)$.

2.1.1 Scenario 1: Forming pre-electoral coalitions

Assume that the partition $\pi = \{S_1, ..., S_J\}$ has formed at the pre-electoral stage and voters have cast their votes. If a coalition S_j - which is possibly a singleton - receives a majority of votes, its members with non zero votes $S' \subseteq S_j$ form a (coalition) government.[16] In the policy selection process, each candidate i in government is recognized as proposal maker with a probability $p_i' > 0$ proportional to the candidate's vote share relative to the other candidates in government. This modeling approach has indirect empirical support and may be motivated by a decision making process in cabinet.[17] In the event where a candidate is recognized, they are free to propose a policy. If the proposal is rejected or no candidate runs, a default outcome is realized where every citizen or candidate realizes a cost $U_i(x^0) = -M < U_i(x)$ for all $x \in [0, 1]$.[18] Thus, every proposal is accepted and rationality dictates that the recognized candidate proposes their ideal point $\widehat{x}_i$.

If no candidate or coalition of candidates receives an overall majority, a policy is negotiated in parliament: we assign $S' = \cup_j S_j$ and assume that every candidate in S' has a probability of being selected proposal maker in proportion to their vote share. If selected, the candidate makes a take-it-or-leave it offer against the default outcome. This policy selection process is compatible with the formation of a minority government (see Pech, 2004) where failure to obtain a majority for its proposal results in the dissolution of parliament, inflicting a cost on all its members.

2.1.2 Scenario 2: Negotiation over policy in the absence of pre-electoral coalitions

In this scenario no pre-electoral coalitions form so the policy is negotiated in full parliament after the election. Formally, we can define a partition $\pi = \{S\}$ as a division of N into those potential candidates who run and are assigned to the broad coalition S, and those potential candidates who do not run and are assigned to the complementary coalition D which we suppress in the representation of π.

If no candidate receives a majority of the votes, each candidate in S with non-zero

[16]For an overview of rules of government formation see Diermeier et al (2003).

[17]Diermeier and Merlo (2004) find empirical support for a proposer selection model which determines the allocation of government departments and where the selection probability of the proposer is proportional to the vote share of a party in government. Laver et al (2011) find a direct proportionality between the allocation of government departments and vote shares, rejecting any formateur advantage in the government formation process. Our proposal maker model focuses on policy selection rather than portfolio allocation and, while the proposal maker has substantial power, from an ex-ante perspective, this power is distributed proportionally.

[18]This assumption ensures that there is always one candidate who wants to run for election, see also Banks and Duggan (2001) or Levy (2004).

vote is included in $S' \subseteq S$ and participates in the negotiations. Each member of S' may be selected as proposal maker with a probability p'_i equal to their vote share p_i to propose against a default outcome x^0 where each candidate realizes $U_i = -M$. Hence, candidate i rationally proposes ideal point $\widehat{x}_i$. If one candidate i wins a majority in the elections, the candidate forms the government $S' = \{i\}$ and implements the preferred policy $\widehat{x}_i$.

2.2 Voting

For our two basic scenarios, we limit rationality at the voting stage and assume that in each scenario, citizens vote sincerely for the ideologically closest candidate, that is the candidate whose ideal point - and equilibrium proposal - is closest to their own ideal point. Claim 1 is useful in establishing our results in the case of scenario 2 where post-electoral negotiations take place in full parliament:

Claim 1. Consider scenario 2 with sincere voting. If the potential candidate preferred by the median voter, m_P, runs for election, this candidate is the median in parliament, i.e. the candidate who turns a minority into a majority when aggregating candidate weights from below or above.

Proof. See part 6.1 of the appendix. □

2.3 Generalized payoff function

Scenarios 1 and 2 make rather restrictive assumptions on the parliamentary selection process and on voting behavior. Here we introduce a more general closed-form function φ that uniquely assigns to each partition the policy lottery which obtains when candidates realize this partition. We postulate that φ represents the results of the policy selection and voting stage. As we show below, the two scenarios introduced above give rise to special cases of this more general form. We state the main result of this paper in terms of the more general function.

Assume that there uniquely exists[19] a mapping $\varphi : \Pi \to L$, where L is the set of lotteries on $[0, 1]$ with the following properties:

Assumption 1. (a) In the case where no candidate runs, i.e. $\pi = \emptyset$, $\varphi(\emptyset) \neq (\widehat{m}_P, 1)$,

[19] Existence and uniqueness is fulfilled for scenarios 1 and 2 introduced above. However, by fixing beliefs or restricting the domain, the arguments of this paper extend to more complex voting and policy selection models which, for example, include a status quo policy point such as in Cho (2004b).

(b) if a candidate or a coalition of candidates receives a majority of votes, they implement their preferred policy or, in the case of a coalition, the policy lottery ℓ associated with its membership, with certainty;

(c) a proposal x' has positive probability in ℓ only if candidates i and k (where possibly $i = k$) with $\widehat{x}_i \geqslant x'$ and $\widehat{x}_k \leqslant x'$ are included in the winning coalition and receive a positive vote share;

(d) let $x'' = \widehat{x}_k$ be the policy that a single candidate k elected for government can implement. Then the policy lottery $\ell = \{(x_i, p'_i)\}_{i \in S'}$ that a coalition $S' \neq \{k\}$ elected for government can implement satisfies $\sum_{i \in S} p'_i (x_i - x'')^2 > 0$;

(e) if the candidate preferred by the median voter runs as a singleton and unopposed on one side of her ideal point, the candidate wins the election.

Property (a) requires that the default outcome that is implemented if no candidate runs for election is different from what m_P would like to implement, so m_P has an incentive to run for a sufficiently small cost of running c.[20] Property (b) states that the election winner is going to implement the preferred policy, which may be a lottery in the case of coalition government. Property (c) states that any proposal must be supported by a candidate (or a coalition of candidates) willing to implement it. For example, a proposal on the right-hand side of the position of the median voter's preferred candidate requires that a candidate with ideal point equally far or further on the right is included in the winning coalition. This includes the case where a candidate is not able to realize their ideal point even when selected because the candidate is constrained when proposing against some status quo point such as in Cho (2014b). Property (d) implies that the certain policy point $(x'', 1)$ can only be implemented if a candidate with ideal point $\widehat{x} = x''$ wins as a singleton. Thereby, we assume that coalition government always involves some policy uncertainty and, hence, implementing a policy point with certainty can only be achieved by the candidate with the same ideal point. We relax this assumption for the case of perfect commitment. By property (e), candidates with ideal points on one side of the median voter's preferred candidate can enforce the outcome where m_P implements her preferred policy $\widehat{m}_P$.

Assumption 1 (e) is compatible with sincere voting behavior but also with sincerely rational voters who vote for the candidate whose equilibrium proposal - if elected - is closest to their own ideal point.[21] It is, in general, violated for strategic

[20]This is weaker than the assumption about default policies in the two scenarios.

[21]See Cho, 2014b. In the case of scenarios 1 and 2, sincerely rational voting coincides with the predictions of sincere voting.

or individually rational voter behavior: in our model where we assume a continuity of voters, no single voter can change the outcome and arbitrary voting strategies are compatible with rationality.[22] In section 4 we consider a refinement where voting strategies form a strong Nash equilibrium. Strong Nash equilibrium may not exist for all policy selection mechanisms and all partitions but, as we demonstrate, for scenario 2, m_P wins a majority in every partition where m_P runs provided that voting strategies form a strong Nash equilibrium.

We can show that scenarios 1 and 2 introduced above satisfy assumption 1:

Claim 2. The mappings of outcomes $\varphi(\pi)$ as determined by scenario 1 and scenario 2 satisfy assumption 1.

Proof. See part 6.2 of the appendix. □

2.4 Party and coalition formation

What a coalition T can do at the coalition formation stage at some status quo partition π is represented by the effectiveness relation $\pi \xrightarrow{T} \pi'$ (see, e.g., Chwe, 1994). In moving from π to π', members of $T \subseteq N$ may join the race, exit the race by joining D and - in the case of pre-electoral coalition formation - consensually form $S_j \subseteq T$ or leave any coalition of which they are member. If T moves to π', π' becomes the new status quo point at which another coalition may move. We derive dominance relations $\succ_T$ on Π from the candidates' preferences on lotteries and for coalition T, $\pi' \succ_T \pi$ if $U_i(\varphi(\pi')) > U_i(\varphi(\pi)) \forall i \in T$. N, Π, the set of preferences U_i and effectiveness relations $\{\xrightarrow{T}\}$ define a game $\Gamma = (N, \Pi, \{U_i\}_{i \in N}, \{\xrightarrow{T}\}_{T \subseteq N, T \neq \varnothing})$.

2.5 Stable coalition structures

We assume that when a coalition considers a move, its members are farsighted, i.e. they anticipate incentives of subsequent coalitions to move. A partition π' may be reached from partition π if it indirectly dominates π, that is if every deviating coalition T_t in a sequence leading from π to π' unanimously prefers π' to the partition π_t from which it deviates.

Definition 1. π is indirectly dominated by π', or $\pi \ll \pi'$, if there exists a sequence $\pi_t \xrightarrow{T_t} \pi_{t+1}$, $t = 0, .., \tau - 1$ with $\pi_0 = \pi$ and $\pi_\tau = \pi'$ such that $\pi' \succ_{T_t} \pi_t$ for all T_t.

[22] See Myerson and Weber (1993) for discrete examples of voting equilibria where voters could jointly improve by coordinating their strategies.

If we also assume that agents are not only farsighted but also optimistic, that is they are willing to deviate from a partition π if at least one of the partitions which may be reached after a deviation leaves them better off, an appropriate solution concept is farsighted stable set $(\Pi, \ll)$ (see, e.g. Chwe, 1994, Xue, 1998):

Definition 2. V is a stable set for $(\Pi, \ll)$ if (1) V is farsighted internally stable: for all π, $\pi' \in V$, neither $\pi \ll \pi'$ nor $\pi' \ll \pi$, and (2) V is farsighted externally stable: $\pi \in \Pi \backslash V$ implies that there is $\pi' \in V$ and $\pi \ll \pi'$.

The farsighted stable set is internally stable, that is free of contradictions: no farsighted optimistic coalition wants to deviate to another partition in V. It is also externally stable, that is it accounts of all partitions which it excludes: a partition is excluded from V only if a farsighted optimistic coalition wants to deviate from it to a partition in V.

Example 1 illustrates the indirect domination relation for the two scenarios for the borderline case where the game is symmetric and, therefore, m's - and m_P's - payoffs from different partitions are not strictly ranked. A typical partition is $\pi' = \{\{1\}, \{2, 3\}\}$ where 1 runs as a singleton and 2 and 3 have formed a coalition.

Example 1. $N = \{1, 2, 3\}$, $\widehat{x}_1 = 0.1$, $\widehat{x}_2 = 0.5$, $\widehat{x}_3 = 0.9$ with $c \geqslant 0$ and default payoff $U_i(x^0) < -1 - c$. Voters' ideal points are equally distributed on $[0, 1]$ and the vector of votes if all candidates run is $(0.3, 0.4, 0.3)$.[23]

Consider scenario 1. It is straightforward to show that $\{\{1, 2, 3\}\} \ll \{\{1\}, \{2\}\}$ via the sequence $\{\{1, 2, 3\}\} \xrightarrow{\{3\}} \{\{1, 2\}\} \xrightarrow{\{2\}} \{\{1\}, \{2\}\}$. Other indirect dominance relations are $\{\{1, 2\}, \{3\}\} \ll \{\{1\}, \{2\}\}$ and $\{\{1\}, \{2\}\} \ll \{\{2, 3\}\}$. Because 2's preferences are symmetric, we neither have $\{\{1, 2\}\} \ll \{\{2, 3\}\}$ nor vice versa. If $c > 0$, agents who are not included in the winning coalition want to exit the race, so the sequence above can be extended to support the relation $\{\{1, 2, 3\}\} \ll \{\{2\}\}$.

Consider scenario 2. For $c = 0$, $\{\{1, 2, 3\}\} \ll \{\{1, 2\}\}$ via the sequence $\{\{1, 2, 3\}\} \xrightarrow{\{3\}} \{\{1, 2\}\}$ where 2 wins a majority. For $c > 0$, we also have $\{\{1, 2, 3\}\} \ll \{\{2\}\}$ as candidate 1, who is not decisive, wants to exit the race.

3 Results

Our results are predicated on coalition preferences on lotteries. Banks and Duggan (2006) have established majority preferences on lotteries for the case of quadratic

[23] For details, see section 7.1 in part 2 of the appendix.

utility functions:[24]

Lemma 1. *(Banks and Duggan, 2006) If candidate i has preference $U_i(\ell_1) > U_i(\ell_2)$ then either all $k < i$ or all $k > i$ have preference $U_k(\ell_1) > U_k(\ell_2)$.*

Proof. See part 6.3 of the appendix. □

The following result on the comparison of lotteries with the certain outcome follows immediately from lemma 1: if a candidate with ideal point on the right-hand side of the ideal point of the candidate preferred by the median voter, $\widehat{m}_P$, weakly prefers a lottery over having $\widehat{m}_P$ with certainty, an agent with a ideal point on the left-hand side of the median voter cannot possibly weakly prefer the same lottery over the certain outcome $\widehat{m}_P$.

Lemma 2. *Let $(\widehat{m}_P, 1)$ be the lottery where the ideal point of m_P is realized with certainty and let $\ell(S)$ be the policy lottery associated with coalition $S \neq \{m_P\}$. If $U_i(\ell) \geqslant U_i(\widehat{m}_P)$ for a candidate i with $\widehat{x}_i > \widehat{m}_P$, then $U_i(\ell) < U_i(\widehat{m}_P)$ for all candidates with ideal point $\widehat{x}_k < \widehat{m}_P$.*

Proof. See part 6.4 of the appendix. □

By lemma 2 there is, for any partition π where m_P does not win the election, a sequence of successive deviations which ultimately bring about a partition where m_P wins and which indirectly dominates π:

Lemma 3. *Let Π_m be the set of all partitions where m_P runs as a singleton and wins the election and let π_m be the partition where all candidates except m_P have exited the race. (a) In the case $c = 0$, for any partition $\pi \in \Pi \backslash \Pi_m$ there exists $\pi' \in \Pi_m$ such that $\pi \ll \pi'$. (b) In the case $c \in (0, \epsilon)$ with $\epsilon > 0$ and sufficiently small, for any partition $\pi \in \Pi \backslash \{\pi_m\}$, $\pi \ll \pi_m$.*

Proof. See part 6.5 of the appendix. □

In partitions in Π_m, the candidate preferred by the median voter wins as a singleton but other candidates may run as well. As the lemma states, for $c = 0$, partitions in Π_m dominate everything else.[25] It is easy to see that partitions in Π_m do not

[24] Beyond quadratic preferences, there are no general results on conditions under which agents agree or disagree on their ranking of different lotteries on the policy line: Banks and Duggan (2006) show that it is possible for two agents on opposite sides of the median to favor a policy lottery which the median voter rejects.

[25] This dominance relation is also direct, although for this to be relevant agents need to be able to device complex agreements on their actions. The uniqueness result in the theorem depends on an indirect dominance relationship.

dominate each other. For $c > 0$ and sufficiently small, π_m, the partition where only m_P runs, dominates everything else as all agents other than m_P want to drop out of the race.

Theorem 1. *Consider a political coalition formation game under proportional representation with citizen candidates and policy correspondence $\varphi(\pi)$. For $c > 0$ and sufficiently small, in the unique farsighted stable set only the candidate preferred by the median voter runs for election and wins. For $c = 0$ and if m_P's preference order on lotteries induced by φ, $L(\varphi, \Pi)$, is strict, the unique farsighted stable set consists of all partitions in Π_m, i.e. where m_P wins a majority.*

Proof. See part 6.6 of the appendix. □

The following application to the situation in example 1 illustrates the theorem and shows that the condition that in the case $c = 0$, m_P's preferences on lotteries are strict, can cannot be suspeneded:

Example. 1. (cont.) In example 1, the unique farsighted stable set for $c > 0$ and and sufficiently small is $V = \{\pi_m\}$ with $\pi_m = \{\{2\}\}$.

For $c = 0$, there are two farsighted stable sets: $V = \Pi_m$ where $\Pi_m = \{\{\{2\}\}, \{\{1\}, \{2\}\}, \{\{2\}, \{3\}\}\}$. And $V' = \{\{\{2\}\}, \{\{1, 2\}\}, \{\{2, 3\}\}\}$. V' is externally stable: $\{\{2\}, \{3\}\} \ll \{\{1, 2\}\}$ via entry of 1, formation of $\{\{1, 2\}\{3\}\}$ and exit of 3. Similarly, $\{\{1\}, \{2\}\} \ll \{\{2, 3\}\}$ via entry of 3 and subsequent exit of 1. Because 2 is indifferent between $\{\{1, 2\}\}$ and $\{\{2, 3\}\}$ none of the corresponding partitions indirectly dominates the other and V' is also internally stable.

With substantial costs of running for election, there are other solutions where the median voter's preferred candidate accepts the policy outcome offered by 1 or 2 while the threat of realizing the default outcome keeps that candidate from exiting.

The farsighted core C^F represents an alternative solution concept for coalition formation processes with farsighted players without the notion of external stability.[26] The farsighted core consists of all farsightedly undominated elements:

$$C^F(\Pi, \ll) = \{\pi \in \Pi | \nexists \pi' \in \Pi \text{ such that } \pi \ll \pi'\}.$$

As π_m is undominated and, for $c \in (0, \epsilon)$, dominates everything else, the following proposition is straightforward:

Proposition 1. *π_m is in the farsighted core of the political coalition formation game under proportional representation with citizen candidates, policy correspondence $\varphi(\pi)$ and $c < \epsilon$. If, in addition, we assume $c > 0$, $C^F(\Pi, \ll) = \{\pi_m\}$.*

[26] See, e.g., Diamantoudi, Xue (2003).

Proof. See part 6.7 of the appendix. □

While making the same predictions as farsighted stable set in the case of small but positive costs c and noting that by lemma 3 a partition $\pi \notin \Pi_m$ cannot be farsightedly undominated and, hence, cannot be in the farsighted core, the following example shows that a partition may be in the farsighted stable set according to theorem 1 but not in the farsighted core $C^F(\Pi, \ll)$.

Example 2. Consider the case of pre-electoral coalition formation in scenario 1 with $N = \{1, 2, 3\}$, $\widehat{x}_1 = 0$, $\widehat{x}_2 = 0.4$, $\widehat{x}_3 = 1$ and $c = 0$ with a vector of votes if all candidates run of $(0.3, 0.4, 0.3)$ and default payoff $U_i(x^0) < -1$.

Assume that the status quo is $\{\{1\}, \{2\}\}$ with 2 winning. Then $\{\{1\}, \{2\}\} \ll \{\{1\}, \{2, 3\}\}$ via the sequence $\{\{1\}, \{2, 3\}\} \xrightarrow{\{3\}} \{\{1\}, \{2\}, \{3\}\} \xrightarrow{\{2,3\}} \{\{1\}, \{2, 3\}\}$.[27] $\{\{1\}, \{2\}\}$ is stable according to theorem 1 but, as it is dominated, it is not in the farsighted core.

Finally, we can ask about the relationship between the set of Nash equilibria and the median voter outcome. For $c > 0$ and $V = \{\pi_m\}$, no entry of an individual candidate changes the outcome, hence the median voter outcome is also a Nash equilibrium. This is also true in example 2 for $c > 0$. But slightly changing the example by assuming that the ideal points are $\widehat{x}_1 = 0.1$, $\widehat{x}_2 = 0.4$, $\widehat{x}_3 = 1$ and the vector of votes if all candidates run is $(0.3, 0.21, 0.49)$, we obtain stable $\{\{1\}, \{2\}\}$ which is not a Nash equilibrium: 3 would like to enter.[28] Moreover, for $c \geqslant 0$ Nash equilibrium does not imply stability: under scenario 1, the partition $\{\{1, 2, 3\}\}$ is a Nash equilibrium in examples 1 and 2.[29]

4 Extensions and alternative assumptions

4.1 Fully rational voters

With a continuum of voters, imposing individual rationality does not eliminate any voting strategies. Here we refine voting equilibrium by requiring that voting strategies form a strong Nash equilibrium, that is we consider joint deviations by voters rather than individual deviations.[30] Focusing on scenario 2, we can show that in a

[27] For details see section 7.2 in part 2 of the appendix.

[28] For details see section 7.2 in part 2 of the appendix.

[29] Under scenario 2, 1 or 3 can obtain the median voter outcome with a single deviation and at least one of them prefers this outcome.

[30] See Baron and Diermeier (2001). A combination of voting strategies forms a strong Nash equilibrium if there is no joint deviation of agents by which these agents can increase their payoff.

strong Nash equilibrium, the candidate preferred by the median voter wins a majority of votes when this candidate runs:

Proposition 2. *Assume that the policy selection mechanism in scenario 2 applies but that voters are rational. For all π where m_P runs, in a strong Nash equilibrium m_P wins the election. Moreover, assume that voting strategies for any given partition are restricted to form a strong Nash equilibrium: then in every partition in the farsighted stable set, m_P runs and wins for ϵ sufficiently small.*

Proof. See part 6.8 of the appendix. □

With the pre-electoral coalition formation mechanism of scenario 1, strong Nash equilibrium predicts voting behavior which is compatible with the observations of Bargsted and Kedar (2009): assume that a coalition S is predicted to form the government. Then, in a strong Nash equilibrium each citizen votes for the member of S with the closest ideal point. Yet, our results also suggests that a coalition government is not going to form in a farsighted stable set.[31] We do not fully analyse this scenario because in a situation where the vote simultaneously determines the winning coalition and also the weights of the members of the winning coalition, a voting equilibrium that is strong Nash may not exist for some partitions π. But if such a case arises, $\varphi(\pi)$ fails to assign a lottery.[32]

4.2 Conservative agents

Our median voter result hinges on optimism as an assumption underlying farsighted stable set as solution concept: the stable set of $(\Pi, \ll)$ coincides with the optimistic stable standard of behavior (Greenberg, 1990) of an appropriately defined situation (Xue, 1998). Optimistic agents are willing to deviate from a partition if one of the partitions which may be reached following a deviation results in a preferred outcome, i.e. if there is a (stable) partition which indirectly dominates the original partition. Chwe's (1994) largest consistent set (LCS), on the other hand, captures stabilitity for conservative agents: agents are only willing to deviate from a partition if all partitions which may potentially be reached after a deviation are preferred by the deviators. Conversely, an original position is in LCS if all deviations are deterred: that is for each position to which agents may deviate, the position itself or another

[31] We can show that in a strong Nash equilibrium, assumption 1 (e) is fulfilled, i.e. m_P wins when running unopposed on one side of her ideal point, so that candidates can enforce the median voter outcome.

[32] In section 7.3 of part 2 of the appendix we construct an example based on scenario 1 where a voting equilibrium which is strong Nash does not exist.

position in the set which indirectly dominates it, fails to be strictly better for all deviators when compared to the original position.

Definition 3. A set Y is consistent if $\pi_a \in Y$ if and only if $\forall \pi_d, S$, such that $\pi_a \xrightarrow{S} \pi_d$, $\exists \pi_e \in Y$, where $\pi_d = \pi_e$ or $\pi_d \ll \pi_e$, such that $\pi_a \not\prec_S \pi_e$.

The largest consistent set contains all consistent sets Y. We know (Chwe, 1994, proposition 3) that LCS contains the stable set of $(\Pi, \ll)$. In our case, it is straightforward to show that the set where the median voter outcome prevails is consistent, that is $\{\pi_m\}$ in the case of $c > 0$ and Π_m in the case of $c = 0$:

Proposition 3. *The farsighted stable set of the political coalition formation game of theorem 1 is also consistent.*

Proof. See part 6.9 of the appendix. □

Generally, $V(\Pi, \ll) \subseteq LCS(\Pi)$. We can give a full characterization of LCS for the case of $n = 3$ where we focus on scenario 1. Let $L(\Pi)$ be set of lotteries on Π.

Proposition 4. *Consider the case of pre-electoral coalition formation in scenario 1 with $N = \{1, 2, 3\}$, median 2, cost $c \in [0, \epsilon)$ where the preference order of the median on $L(\Pi)$ is strict. Then $LCS = \Pi_m$ for $c = 0$ and $LCS = \{\pi_m\}$ for $0 < c < \epsilon$.*

To see why this result holds, assume that 2 prefers $\{\{2, 3\}\}$ to $\{\{1, 2\}\}$ and that $\{\{1, 2\}\}$, has formed. Let 2 deviate to $\pi_d = \{\{1\}, \{2\}\}$. The deviation is undeterred as π_d is preferred by 2 and the sequence where 3 enters to form $\{\{1\}, \{2, 3\}\}$ and $\{\{2, 3\}\}$ does not deter 2. So $\{\{1, 2\}\} \notin LCS$. Moreover, $\{\{2, 3\}\} \notin LCS$: if 2 deviates to form $\{\{2\}, \{3\}\}$, 1 would only want to enter if $\{\{1, 2\}, \{3\}\}$ and subsequently $\{\{1, 2\}\}$ may form. But as we have shown, $\{\{1, 2\}\} \notin LCS$ and, therefore, does not deter the deviation.[33]

But while we obtain a median voter result for the asymmetric case, the following example demonstrates that in the symmetric case where 2 is indifferent between the partitions $\{\{1, 2\}\}$ and $\{\{2, 3\}\}$, both partitions are in LCS:

Example 3. Consider the case of pre-electoral coalition formation in scenario 1 with $N = \{1, 2, 3\}$, median 2, $0 \leqslant c < \epsilon$, and assume that no candidate wins a majority in $\{\{1\}, \{2\}, \{3\}\}$. Moreover, 2 is indifferent between $\{\{1, 2\}\}$ and $\{\{2, 3\}\}$. Then, apart from $\Pi_m \subseteq LCS$ in the case $c = 0$ and $\{\pi_m\} \in LCS$ in the case $0 < c < \epsilon$, we also have $\{\{1, 2\}\}, \{\{2, 3\}\} \in LCS$.

[33] A full discussion of the details is included in section 7.4 in part 2 of the appendix.

To see why LCS is multi-valued for $c > 0$, assume that $\{\{2,3\}\}$ has formed. Consider a deviation by 2 where $\pi_d = \{\{2\},\{3\}\}$ forms. This deviation is deterred: the sequence where 1 enters to obtain $\pi_d = \{\{1\},\{2\},\{3\}\}$ and, subsequently $\{\{1,2\},\{3\}\}$ and $\pi_e = \{\{1,2\}\}$ form indirectly dominates and leaves 2 no better off than $\{\{2,3\}\}$. Similarly, a deviation by 2 from $\{\{1,2\}\}$ is deterred by the threat that $\{\{2,3\}\}$ may form. So $\{\{1,2\}\},\{\{2,3\}\} \in LCS$.

Comparison between the symmetric example 1 (cont.) and example 3 shows that for $c = 0$, LCS contains the stable sets V and V'. For $0 < c < \epsilon$, however, the stable set $V = \{\{2\}\}$ is unique while LCS continues to comprise $\{\{2\}\}$, $\{\{1,2\}\}$ and $\{\{2,3\}\}$. So conservatism expands the set of predicted policies and supporting coalitions.

4.3 Perfect commitment

The possibility of commitment only arises in the case of pre-electoral coalition formation. In line with assumption 1 (c), we assume that at the time when a coalition S forms it can commit to implement any policy point x_S within the convex set contained in $[\widehat{x}_{S_l}, \widehat{x}_{S_u}]$, the range between the ideal points of its left-most and right-most coalition member.[34] Moreover, we assume that voters are rationally sincere in that they vote for some candidate in the coalition whose policy proposal they prefer and that they break indifference between the candidates forming the coalition by voting for that coalition member with the preferred ideal point. An outcome of the coalition formation game now is a pair $(\pi, \mathbf{x})$ where $\mathbf{x}$ is a vector of policy announcements. We refer to $(\pi, \mathbf{x})$ as a position. If no candidate wins an outright majority we assume that the outcome is a lottery on $\mathbf{x}$, in the case where there is a majority winner x_S, we indicate the winner in the notation of the position as $(\pi, x_S; \mathbf{x}_{-S})$.

We have to define effectiveness relations for this game. As before, agents may leave the race, they may form new coalitions and they may also enter the race. When a coalition T forms it announces a policy $x^T \in [\widehat{x}_{T_l}, \widehat{x}_{T_u}]$ of which all its members agree when compared to the position from which it departs. If members of T were indispensable for supporting the policy announcement, x_S, of any coalition S from which they split, we assume that the policy associated with S also changes. This case only can occur when T includes the previous boundary of S and $x_S \in [\widehat{x}_{T_l}, \widehat{x}_{T_u}]$. After departure of those agents which join T, the announcement x_S is no longer credible. As S had wished to implement x_S when this was feasible, we assume that the credible policy x'_S associated with $S \backslash T$ is at the new boundary of $S \backslash T$ which is

[34] Essentially, this assumption restricts policy announcements to the set of policies which are Pareto-undominated for the coalition. This may be seen a credibility constraint. See Levy (2004) for a similar assumption on policies which parties can implement.

now becoming binding.[35]

Definition 4. The effectiveness relation $(\pi, \mathbf{x}) \xrightarrow{T} (\pi', \mathbf{x}')$ implies that $\pi' = \{\pi \backslash T, T\}$ and $\mathbf{x}' = \mathbf{x}$ except for x_T which is added to $\mathbf{x}'$ and any policy announcement x_S violating $x_S \in [\widehat{x}_{S_l'}, \widehat{x}_{S_u'}]$ for $S' = S \backslash T$ which is replaced in $\mathbf{x}'$ by $x'_S = \widehat{x}_{S_l'}$ if $\widehat{x}_{S_l} \leqslant x_S \leqslant \widehat{x}_{S_l'}$ or $x'_S = \widehat{x}_{S_u'}$ if $\widehat{x}_{S_u} \geqslant x_S \geqslant \widehat{x}_{S_u'}$.

The following proposition shows that under perfect commitment, the predicted policy coincides with the policy predicted by the theorem. However, unlike in the theorem, for $c = 0$ any coalition which extends to both sides of the median may offer this policy.

Proposition 5. *Assume that a coalition S can commit to implement a policy $x_S \in [\widehat{x}_{S_l}, \widehat{x}_{S_u}]$ conditional on winning the election. For $c = 0$, in every $(\pi, \mathbf{x})$ in the unique farsighted stable set V a coalition committed to implementing the outcome $\widehat{m}_P$ prefered by the median voter's favored candidate m_P wins. For $0 < c < \epsilon$ with ϵ sufficiently small, there exists a stable set $V = \{(\pi_m, x_{\{m_P\}} = \widehat{m}_P)\}$ where only the candidate preferred by the median voter runs.*

Proof. See part 6.10 of the appendix. □

For $c = 0$, all partitions with coalitions that support the outcome $\widehat{m}_P$ are in the stable set. For $c > 0$ but sufficiently small, positions with different coalitions offering $\widehat{m}_P$ dominate each other, as each coalition would like to avoid incurring a cost of c. $(\pi_m, x_{\{m_P\}} = \widehat{m}_P)$ dominates all other partitions and, therefore, is always stable. Stable sets in which other coalitions support the outcome $\widehat{m}_P$ may exist, as the following continuation of example 2 shows:[36]

Example. 2. (cont.)
In example 2, for $c = 0$ with 2 the candidate preferred by the median voter, the unique farsighted stable set is:
$V = \{(\{\{2\}\}, x_{\{2\}} = \widehat{m}_P), (\{\{1,3\}\}, x_{\{1,3\}} = \widehat{m}_P), (\{\{1\},\{2\}\}, x_{\{2\}} = \widehat{m}_P), (\{\{1\},\{2\}\}, x_{\{2\}} = \widehat{m}_P), (\{\{1,2\}\}, x_{\{1,2\}} = \widehat{m}_P), (\{\{2,3\}\}, x_{\{2,3\}} = \widehat{m}_P), (\{\{1,2,3\}\}, x_{\{1,2,3\}} = \widehat{m}_P)\}$.
For for $c > 0$ and sufficiently small and $U_i(x^0) < -1 - c$ for all i, there is $V = \{(\pi_m, x_{\{m_P\}} = \widehat{m}_P)\}$ and $V' = \{(\{\{1,3\}\}, x_{\{1,3\}} = \widehat{m}_P), (\{\{1,2\}\}, x_{\{1,2\}} = \widehat{m}_P), (\{\{1\},\{2\}\}, x_{\{2\}} = \widehat{m}_P), (\{\{2,3\}\}, x_{\{2,3\}} = \widehat{m}_P)\}, (\{\{2\},\{3\}\}, x_{\{2\}} = \widehat{m}_P)\}$.

[35]The selection of x'_S is plausible in that it Pareto-dominates x_S for the remaining members of S.
[36]Details of this example are provided in Appendix 2

Intriguingly, while our theory still predicts the outcome $\widehat{m}_P$ of the candidate preferred by the median voter, this outcome may be implemented by different coalitions with diverging policy goals, even when the cost of running for election is positive. Because of the cost of running, each supporting coalition would prefer another coalition to implement this outcome, so the stable set is typically non unique.

4.4 Comparison with Plurality Rule

In order to compare our results on PR systems with first-past-the-post systems we propose a simple multi-constituency model: there is an odd number Z of constituencies, $R_1, ..., R_Z$, and each constituency elects one candidate to parliament by plurality rule. Unlike in an ideal PR system where there is just one "constituency" which coincides with the nation state and where a candidate can represent a party (or a party list), a multi-constituency model forces us to explicitly model a parties as multitudes of actors. For simplicity, we assume that for each of the n candidates - which we may identify as parties - there exist Z clones, so that each candidate or party may field up to Z identical clones - one in each constituency. Parties - represented by the cloned candidates - can form pre-electoral coalitions or bargain over goverment after elections. Parliament decides on a policy x which affects all constituencies. Citizens including candidates have preferences on x.

A partition now is a division of cloned candidates in Z constituencies who run or exit the race in any constituency. To avoid issues resulting from trading votes across constituencies, we assume that voters in each constituency sincerely vote for the ideologically closest (clone of a) candidate running in that constituency. Moreover, we assume that the policy selection procedure in parliament follows scenarios 1 or 2, that is a pre-electoral coalition that holds a majority in parliament may put a proposal from a randomly selected coalition member (scenario 1) against a default outcome or a party that holds a majority in parliament may put a proposal (scenario 2) or, if there is no majority in parliament, any candidate may be selected as proposal maker with a probability proportional to their weight in parliament. Let $\widehat{\varphi}(\pi)$ be a mapping of a partition of (cloned) candidates in the Z constituencies into seats in parliament and seats in parliament into policy outcomes.

To capture the cost of running for election we consider two alternative cost functions: in the first, the cost of running for a party i is fixed at C once the candidate or party fields a clone in any constituency. This captures the case where a central party office provides a public good for all its constituencies where it fields a clone. In the second, the cost for party i of running is proportional to the number of constituencies z_i in which the candidate or party decides to field a clone. In the case where cost is proportional, there are incentives for a party to reduce its parliamentary representation to the smallest possible number which gives it a majority - which may be as small as one if no other party is running. Here, our result hinges on the imposition of a quorum which parliament needs to meet in order to adopt a policy. The following two alternative assumptions more fully characterize the scenarios for which we derive our results:

Assumption 2. The cost of running for election $C(z_i)$ for any candidate or party i satisfies $C(0) = 0$ for $z_i = 0$ and $0 < C(z_i) < \epsilon$ for $z_i > 0$.

Assumption 3. For candidate or party i, the cost of running for election is $C(z_i) = \epsilon z_i$, $\epsilon > 9$, and there is a quorum in parliament which requires $\frac{Z+1}{2}$ constituency representatives to vote for a propposal for it to be adopted.

The following lemma parallels lemma 3: a partition where the party m_P preferred by the median of the median constituency, m_ζ runs in enough constituencies to get a parliamentary majority indirectly dominates all partions where a party other than m_P is winning and also all partitions where m_P is winning but other parties are also running:

Lemma 4. *Assume that the policy selection correspondence is $\widehat{\varphi}$ and that either assumption 2 or 3 holds. Let $\widehat{\Pi}_m$ be the set of all partitions where only the party m_P preferred by the median voter of the median constituency m_ζ runs in $\frac{Z+1}{2}$ constituencies and let $\widetilde{\Pi}_m$ be the set of partitions where only m_P runs in $z_m > 0$ constituencies. If ϵ is sufficiently small, for every partition $\pi \in \Pi \backslash \widetilde{\Pi}_m$ there exists $\pi' \in \widehat{\Pi}_m$ such that $\pi \ll \pi'$.*

Proof. See part 6.11 of the appendix. □

The unique farsighted stable set consists of partitions where where only m_P is running. In the case with proportional cost and a quorum of $\frac{Z+1}{2}$ the stable set coincides with all partitions where m_P just reaches the quorum. In the case where the cost of running is fixed rather than proportional, the set of partitions where only m_P runs is stable:

Proposition 6. *In the multi-constituency model with plurality voting and with policy correspondence $\widehat{\varphi}$, let m_ζ be the median of constituency medians in terms of the policy line x, let m_P be the party most preferred by m_ζ. If (a), assumption 2 holds, for ϵ sufficiently small, the unique farsighted stable set of the multi-constituency model with plurality voting consists of all partitions $\widetilde{\Pi}_m$ where only m_P runs in $z_m > 0$*

constituencies. If (b) assumption 3 holds and for ϵ sufficiently small, the unique farsighted stable set consists of all partitions $\widehat{\Pi}_m$ where only m_P runs in $z_m = \frac{Z+1}{2}$ constituencies.

Proof. See part 6.12 of the appendix. □

In the case of assumption 2 where the cost is incurred upon entry but does not vary in the number of contested constituencies, all partitions where the party preferred by the median voter runs in some number $z_m > 0$ of constituencies - i.e. in $\widetilde{\Pi}_m$ - are payoff equivalent and included in the farsighted stable set. Although choosing some particular $z_m < \frac{Z+1}{2}$ appears vulnerable to entry, the argument is that any challenge by another party would be blocked by some element in $\widetilde{\Pi}_m$ and, therefore, is deterred. If the default outcome is sufficiently small, there is no deviation which aims at forming a government which does not meet the quorum in assumption 3.

5 Conclusion

This paper establishes a median voter result for a model of proportional representation with citizen candidates who are policy motivated and face either small or zero costs of running for election: in the farsighted stable set of a political coalition formation game, the candidate preferred by the median voter runs as a singleton and wins the election. What drives this result is that bargaining in coalitions is associated with policy lotteries, while only single candidates can implement a single policy point, and citizen-candidates dislike policy uncertainty. Our fundamental result holds for a range of policy selection mechanisms. We introduce two scenarios to illustrate such a mechanism: in one scenario candidates may form firm pre-electoral coalitions but cannot commit to a policy point. In the other scenario policies are negotiated in full parliament after the election. Voters in the two illustrative scenarios are voting sincerely, but at least for the case of policy negotiations in parliament a median voter result applies when voting strategies form a strong Nash equilibrium.

To obtain the media voter result, some more specific assumptions are necessary: citizens' and candidates' utility is quadratic, the costs of running for election are negligibly small and in the case of zero cost of running for election, the candidate preferred by the median voter has strict preferences on the set of feasible policy lotteries. Moreover, the use of farsighted stable set as a solution method implies that candidates are farsighted and optimistic. That is, candidates are willing to deviate from a position if there exists one stable position which indirectly dominates the original position and, hence, may be reached by a sequence of deviations.

Relaxing some of these assumptions increases the set of predicted outcomes: to capture conservative behavior on the side of citizen candidates, we derive the largest consistent set of a political coalition formation game. While the median voter result continues to be a solution for conservative citizen candidates, other outcomes may be supported. If a (pre-electoral) coalition can commit to implement a policy in the Pareto-set of its coalition members, the predicted policy outcome coincides with the ideal point of the candidate preferred by the median voter, but different partitions support this outcome: in the case of zero costs of running for elections this outcome is supported by different coalitions which may form in the unique farsighted stable set. In the case of positive but neglible costs, different coalition form to support this outcome in different stable sets.

In order to investigate the impact of the voting institution, we apply the fundamental modeling assumptions to a multi-constituency model where each constituency elects a clon of a candidate by plurality rule. While the cost function - fixed or proportional to the number of clones which a candidates fields - plays a role for how many constituencies are candidate contests, the elementary median voter result also carries over to this variant of the model. Here, the conclusion is that the median voter result obtains irrespective of the voting institution.

So while the innovation of this paper is to claim a median voter result for proportional representation systems, its implication is a convergence of policy outcomes and competing candidates under proportional representation systems and parliamentary first-past-the-post systems. So while our framework leaves the empirically observed divergence of policy outcomes and also the difference in the number of effective parties under each voting system (Leijphart, 1999) to be explained, it provides a common point of departure for the analysis.

Our model ignores non instrumental voting behavior: Kedar (2005) argues that voters may vote for parties who put an issue on the agenda even if they do not prefer the policy position of the party. Candidates may seek office rather than a policy outcome or they may be ideologues who do not wish to compromise. They may exert influence also when they are not represented in government and maybe not even in parliament (Huber and Powell, 1994). Moreover, the paper ignores centrifugal forces in plurality voting systems and - with the exception of the model variant with commitment - it also ignores centripedal forces in the multiparty systems which empirically form under PR.[37]

[37] See the discussion in the introduction.

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6 Appendix 1: Proofs

6.1 Proof of claim 1

Let v_i be the votes cast for candidate i. Assume $\widehat{m}_P > \widehat{x}_m$ and let m_{P-1} be the candidate adjacent on the left side of m_P (if there is no such candidate, the claim immediately follows). Because $U(\widehat{m}_P; \widehat{x}_m) > U(\widehat{m}_{P-1}; \widehat{x}_m)$, by continuity of h and symmetry of the utility function there is $\eta > 0$ such that $U(\widehat{m}_P; \widehat{x}) \geqslant U(\widehat{m}_{P-1}; \widehat{x})$ for $\widehat{x} \in (\widehat{x}_m - \eta, \widehat{x}_m)$, hence $\sum_{i=1}^{m_P} v_i > 0.5$ but $\sum_{i=1}^{m_{P-1}} v_i < 0.5$. An analogous relationship holds for m_P and m_{P+1} when aggregating votes from above.

6.2 Proof of claim 2

Part a) of assumption 1 is satisfied by the assumption on the default policy.

Part b) is satisfied: by rationality, a candidate winning as a singleton implements the preferred outcome. In scenario 1, a winning coalition S_k, implements its associated policy lottery $\ell = (\widehat{x}_i, p'_i)_{i \in S_k}$, where $p'_i = \frac{p_i}{\sum_{j \in S_k} p_j}$ and $\sum_i p'_i = 1$. In scenario 2, if there is no winning candidate, the coalition S' of agents who run and receive non zero votes implements the lottery $\ell = (\widehat{x}_i, p_i)_{i \in S'}$ with $\sum_i p_i = 1$.

To show that part c) holds, we observe that in the lottery corresponding to scenario 1, $x \geqslant x'$ has positive probability only if a candidate i with $\widehat{x}_i = x \geqslant x'$ is included S_k and $p'_i > 0$. An analoguous observation holds for scenario 2 and S'.

Part d) follows because only if a singleton coalition $\{k\}$ wins a majority in scenario 1 or one candidate k wins a majority in scenario 2, k proposes her ideal point $\widehat{x}_k$ while in all other cases, candidates in S_k (or S') make a proposal with $p'_i > 0$.

Finally, part e) follows from the assumption that citizens vote for the candidate closest to their ideal point.

6.3 Proof of lemma 1

In the quadratic case, $U_{\widehat{x}}(\ell)$ can be decomposed into into an expression which depends on the mean of the lottery, $g(\ell)$, and its variance, $var(\ell)$, such that the expected utility of an agent with ideal point $\widehat{x}$ can be written as $U_{\widehat{x}}(\ell) = -[g(\ell) - \widehat{x}]^2 - var(\ell)$ and the difference in this agent's utility between two lotteries as $-[g(\ell_1) - \widehat{x}]^2 + [g(\ell_2) - \widehat{x}]^2 - var(\ell_1) + var(\ell_2)$. It follows immediately that if $g(\ell_1) < g(\ell_2)$ and an agent with $\widehat{x}_i$ prefers ℓ_1 over ℓ_2, then all agents with $\widehat{x} < \widehat{x}_i$ must also prefer ℓ_1 over ℓ_2.

6.4 Proof of lemma 2

For m_P, by assumption 1 (c), $U_{m_P}(\widehat{m}_P, 1) > U_{m_P}(\ell)$. By lemma 1, all $i > m_P$ or all $k < m_P$ must have $U(\widehat{m}_P, 1) > U(\ell)$. So assume that $U_i(\widehat{m}_P, 1) \leqslant U_i(\ell)$ for i with $\widehat{x}_i > \widehat{m}_P$. Then it follows that $U_k(\widehat{m}_P, 1) > U_k(\ell)$ for all $k < m_P$.

6.5 Proof of lemma 3

6.5.1 $c = 0$

Let S be set of all coalitions of candidates running for election, i.e. $S = \cup_j S_j$. By assumption 1 (c), $\varphi(\pi) \neq (\widehat{m}_P, 1)$ for $\pi \in \Pi \backslash \Pi_m$.

Assume $S = \varnothing$. As $\varphi(\varnothing) \neq (\widehat{m}_P, 1)$, m_P enters the race for c sufficiently small and forms $\pi_m \in \Pi_m$. So assume that $S \neq \varnothing$. Let s_l be the lower and s_h the upper boundary of S.

(a) If $s_l \geqslant m_P$ or $s_h \leqslant m_P$ but $\pi \notin \Pi_m$, by assumption 1 (e), m_P wins if she enters as a singleton and forms $\pi' \in \Pi_m$, so $\pi \ll \pi'$.

(b) If $s_l < m_P$ and $s_h > m_P$, $\pi \notin \Pi_m$, for $i \in S$ with $i > m_P$, either (aa) $U_i(\ell) < U_i(\widehat{m}_P, 1)$ or (bb) $U_i(\ell) \geqslant U_i(\widehat{m}_P, 1)$. Assume that (aa) holds. If there is also one agent $k < m_P$ with $U_k(\ell) \geqslant U_k(\widehat{m}_P, 1)$, by lemma 2 there is $\pi' \in \Pi_m$ such that $U_i(\varphi(\pi')) > U_i(\varphi(\pi)$ for all $i > m_P$ so by the sequence all i exit and m_P runs as singleton, $\pi \ll \pi'$. If there is no such agent $k < m_P$, there is $\pi' \in \Pi_m$ such that for all agents $k < m_P$, $U_k(\varphi(\pi')) > U_k(\varphi(\pi))$ and $\pi \ll \pi'$. So assume that (bb) holds for some $i \in S, i > m_P$. In this case, by lemma 2, there is $\pi' \in \Pi_m$ such that for all $k < m$, $U_k(\varphi(\pi')) > U_k(\varphi(\pi))$ and $\pi \ll \pi'$.

6.5.2 $c \in (0, \epsilon)$

Assume $c > 0$ and define ϵ as the minimum value of c such that m_P is indifferent between running and implementing $\ell' = (\widehat{m}_P, 1)$ and the higher utility that she receives with (a) the closest candidate i running and implementing $\ell' = (\widehat{x}_i, 1)$ or (b) $\varphi(\emptyset)$. $c < \epsilon$ is a sufficient condition for m_P to replace any partition π where she does not run with a partition π' where $\ell' = \varphi(\pi')$.

So consider a partition $\pi' \in \Pi_m$, $\pi' \neq \pi_m$. For all $\pi \notin \Pi_m$, $\pi \ll \pi'$ by section 6.5.1 and $\pi \ll \pi_m$ as all candidates who run in π' and incur c prefer to exit.

6.6 Proof of theorem 1

6.6.1 The case $0 < c < \epsilon$.

By lemma 3, with $c \in (0, \epsilon)$, for all $\pi \neq \pi_m$, $\pi \ll \pi_m$. Therefore, $V = \{\pi_m\}$ is externally stable. Because V contains only one element, it is trivially also internally stable.

Uniqueness: Suppose that there is $V' \neq V$. Because no stable set contains another, by external stability, there must be $\pi' \in V'$ such that $\pi_m \ll \pi'$. Suppose that there is $i > m_P$ such that $U_i(\varphi(\pi')) > U_i(\varphi(\pi_m))$. But then by lemma 2, $U_k(\varphi(\pi')) < U_k(\varphi(\pi_m))$ for all $k < m_P$. Therefore, no agent $k < m_P$ deviates when the status quo is π_m, m_P does not deviate because $(\widehat{m}_P, 1)$ is maximal for c sufficiently small and, by assumption 1 (e), m_P wins a majority in π', contradicting that $U_i(\varphi(\pi')) > U_i(\varphi(\pi_m))$.

6.6.2 The case $c = 0$.

Let Π_m be the set of partitions where m_P wins a majority. $V = \Pi_m$ is a stable set: by lemma 3, every $\pi \in \Pi \backslash \Pi_m$ is indirectly dominated by an element in Π_m, hence V is externally stable. As all elements in Π_m are associated with the lottery $(\widehat{m}_m, 1)$, no element dominates another element in the set, hence V is internally stable.

We have to show that V is also the unique stable set. The following claim immediately obtains from lemma 1:

Claim 3. Assume that $g(\ell_1) < \widehat{m}_P$ and $g(\ell_2) > \widehat{m}_P$. If $U_{m_P}(\ell_2) > U_{m_P}(\ell_1)$ then $U_i(\ell_2) > U_i(\ell_1)$ for all $i > m_P$.

Suppose there is $V' \neq \Pi_m$. Assume $\overline{\pi}' \in \overline{\Pi}$, where $\overline{\Pi} \subset \Pi_m$ is the set of partitions where m_P is winning as a singleton and all agents $k < m_P$ have exited and, similarly, define $\underline{\Pi}$ where all agents $i > m_P$ have exited. As no stable set contains another, there must be $\overline{\pi}' \in \Pi_m$ and $\pi_1 \in V'$ such that $\overline{\pi}' \ll \pi_1$ and $\pi_1 \in V'$.

By assumption 1 (e), any deviation from $\overline{\pi}'$ must be initiated by $k < m_P$, so that the expected value, g, of lotteries associated with partitions in the set of partitions dominating $\overline{\pi}'$, $\Pi^D(\overline{\pi}')$, must satisfy $g(\varphi(\pi)) < \widehat{m}_P$. Choose π_1 to be the highest element in $\Pi^D(\overline{\pi}')$ with respect to $U_{m_P}(\varphi(\pi))$.

Let $\underline{\pi}_1 \in \underline{\Pi}$ be the partition obtained from π_1 by dropping all agents with $i > m_P$ where m_P runs as a singleton. By lemma 2, $\pi_1 \ll \underline{\pi}_1$. By external stability, $\exists \pi_2 \in V'$ such that $\underline{\pi}_1 \ll \pi_2$. As the move to π_2 must be initiated by $i > m_P$, we must have $g(\varphi(\pi_2)) > \widehat{m}_P$.

(a) Suppose $U_{m_P}(\varphi(\pi_2)) > U_{m_P}(\varphi(\pi_1))$. By claim 3, it follows that for all $i > m_P$, $U_i(\varphi(\pi_2)) > U_i(\varphi(\pi_1))$, and there exists a sequence $\pi_1 \longrightarrow \underline{\pi}_1 \longrightarrow \pi_2$ such that $\pi_1 \ll \pi_2$. By internal stability, this contradicts that $\pi_1 \in V'$.

(b) Suppose $U_{m_P}(\varphi(\pi_1)) > U_{m_P}(\varphi(\pi_2))$.

Obtain $\overline{\pi}_2$ by dropping all $k < m_P$ from π_2. By lemma 2, $\pi_2 \ll \underline{\pi}_2$. By external stability, $\exists \pi_3 \in V'$ such that $\overline{\pi}_2 \ll \pi_3$.

Either a condition corresponding to (a) holds, i.e. $U_{m_P}(\varphi(\pi_3)) > U_{m_P}(\varphi(\pi_2))$, resulting in a contradiction. Or the reverse condition - call it (b)' - holds.[38]

So suppose that (b)' holds. Set $t = 3$, $t + 1 = 4$ and repeat the argument above for increasing values of t and π_t, π_{t+1}. If condition (b)' continues to hold for t and $t + 1$, we have a decreasing sequence. As there cannot exist an infinite decreasing sequence on a finite set Π, one of the following must occur:

(ba) Suppose, the sequence breaks, i.e. that $U_{m_P}(\varphi(\pi_{(t+1)})) > U_{m_P}(\varphi(\pi_t))$. By internal stability, $\pi_t \notin V'$ and for partitions in the sequence in $\underline{\Pi} \cup \overline{\Pi}$, including $\underline{\pi}_1$, we have $\underline{\pi}_s \in V'$ or $\overline{\pi}_s \in V'$, $s < t$, and, hence, $\pi' \notin V'$.

(bb) Suppose that $\pi_{t+1} = \pi_s$ for some $s < t$.

As $U_{m_P}(\varphi(\pi_s)) > U_{m_P}(\varphi(\pi_t)) > U_{m_P}(\varphi(\pi_{t+1}))$ this violates transitivity of m_P's preference order.

We have demonstrated that $\pi_1 \notin V'$. As this argument holds for all $\pi \in \Pi^D(\overline{\pi}')$, by external stability, $\overline{\pi}' \in V'$. Moreover, the same argument can be established for all elements in $\underline{\Pi} \cup \overline{\Pi}$. By lemma 3, together these elements block all $\pi \in \Pi \backslash \Pi_m$. Therefore, $V' = \Pi_m = V$.

[38] Note that we exclude equality.

6.7 Proof of proposition 1

Proposition 1 follows immediately from lemma 3: for $c \in (0, \epsilon)$, all partitions in $\Pi \backslash \pi_m$ are dominated by π_m, and for $c = 0$, all partitions in $\Pi \backslash \Pi_m$ are dominated by some element in Π_m. Inducing $\pi \notin \Pi_m$ from π_m requires entry on both sides of m_P. Therefore, by lemma 2, π_m is not dominated via $\ll$.

6.8 Proof of proposition 2

Assume that m_P runs in π and that the vector of votes is $\mathbf{p}$ with corresponding $\ell \neq (\widehat{m}_P, 1)$. By claim 2, the policy selection mechanism under scenario 2 satisfies assumption 1 (a) - (d), so in $\mathbf{p}$, m_P does not win a majority. By lemma 1, either all voters with $\widehat{x} \leqslant \widehat{x}_m$ or all voters with $\widehat{x} \geqslant \widehat{x}_m$ prefer $(\widehat{m}_P, 1)$ over ℓ. By switching their vote to m_P, these voters can ensure that m_P wins a majority and implements $(\widehat{m}_P, 1)$. Hence, voting strategies in $\mathbf{p}$ do not form a strong Nash equilibrium. Reversing this argument, if x_P wins a majority, there is no deviation to a vector of votes $\mathbf{p}$ where m_P does not win a majority and a majority of agents improve their utility. Note, that different voting strategies support an outcome where m_P wins the election.

To show that in every V, m_P runs, suppose that there is π' where m_P does not run. By claim 2 the corresponding lottery is $\ell' \neq (\widehat{m}_P, 1)$ and for ϵ sufficiently small, m_P enters, forms partition π and, with voting strategies that yield a strong Nash equilibrium, wins the election. Moreover, for $c > 0$, all candidates other than m_P exit the race. So there is $\pi \in V$ such that $\pi' \ll \pi$ and V is externally stable. As all π where m_P wins a majority (and, for $\epsilon > 0$ other candidates have exited) are payoff equivalent, V is also internally stable.

6.9 Proof of proposition 3

Assume that $Y' = \pi_m$ in the case $c \in (0, \epsilon)$ or $Y' = \Pi_m$ in the case $c = 0$. By lemma 3, for all $\pi' \notin Y'$, $\exists \pi \in Y'$ such that $\pi' \ll \pi$. So for every deviation π_d, $\exists \pi_e \in Y'$, where $\pi_d = \pi_e$ or $\pi_d \ll \pi_e$, such that $\pi \not\prec_S \pi_e$. Hence Y' is consistent.

6.10 Proof of proposition 5

6.10.1 The case $c = 0$

Assume $c = 0$ and let $(\pi, x_S; \mathbf{x_{-S}})$ - in short, (π, x_S) or, when we need to label a particular partition, (π_S, x_S) - denote a position π where coalition S with its policy announcement x_S wins when the vector of proposals by all coalitions in π other than S is $\mathbf{x}_{-S}$. We show that V consists of all (π, x_S) with $x_S = \widehat{m}_P$:

Stability: Let $\overleftarrow{S}$ be the set of all candidates $k < m_P$ and $\overline{\Pi}$ the set of partitions where m_P runs as a singleton and $\overleftarrow{S}$ have exited and $\underline{\Pi}$ the set where m_P runs and all $k > m_P$ have exited.

For any (π, x_S) with $x_S \neq \widehat{m}_P$, there is $\pi' \in \underline{\Pi} \cup \overline{\Pi}$ and $(\pi', x_{\{m_P\}} = \widehat{m}_P)$ such that $(\pi, x_S) \ll (\pi', x_{\{m_P\}} = \widehat{m}_P)$: assume $x_S < \widehat{m}_P$ and consider the following sequence: all agents in $\overrightarrow{S}$ exit the race, m_P runs as a singleton and offers $\widehat{m}_P$. The remaining coalition $T \subset S \backslash \overrightarrow{S}$ must offer $x_T \leqslant \widehat{m}_{P-1}$ but then, by definition, m votes in favor of m_P and in the position $(\pi', x_{\{m_P\}} = \widehat{m}_P)$, m_P wins the election. This shows external stability. Internal stability is given as all elements in V offer the same payoff, so no element in V dominates another element in the set. Hence V is internally and externally stable, that is V is stable.

Uniqueness:

Let $\overline{\pi} \in \overline{\Pi}$. As no stable set contains another, we need to block $(\overline{\pi}, x_{\{m_P\}} = \widehat{m}_P)$. Any deviation to change this outcome, i.e. $(\overline{\pi}, x_{\{m_P\}} = \widehat{m}_P) \ll (\pi_S, x_S)$, must be initiated by agents in $\overleftarrow{S}$ and, therefore, $x_S < \widehat{m}_P$.[39] Note that in the sequence from $(\overline{\pi}, \widehat{m}_P)$ to (π, x_S), m_P must lose her majority status, giving rise to a lottery on proposals, ℓ, and, as m_P and agents in $\vec{S}$ need to drop out of the race, (π_S, x_S) must satisfy $U_{m_P}(x_S) > U_{m_P}(\ell)$. Assume that $x_S \leqslant \widehat{m}_P - \delta$.

Suppose $(\pi_S, x_S) \in V'$ and assume that $\widehat{m}_P \leqslant \widehat{x}_m$. $(\pi_S, x_S) \ll (\pi_T, x_T)$ with $T = \{m_P, m_{P+1}\}$ and $x_T < \widehat{m}_P + \delta$. So we need to ensure that $(\pi_T, x_T) \notin V'$. There are two cases:

(a) Suppose $(\pi_T, x_T) \ll (\pi'_S, x'_S)$ where $(\pi'_S, x'_S) \in V'$ and $x'_S < \widehat{m}_P$, satisfying $|x'_S - \widehat{m}_P| < |x_T - \widehat{m}_P|$.[40] But as all agents $i \geqslant m_P$ prefer x'_S over x_S, $(\pi_S, x_S) \ll (\pi'_S, x'_S)$ via $(\pi_S, x_S) \xrightarrow{T} (\pi_T, x_T)$, contradicting that $(\pi_S, x_S) \in V'$.

(b) Suppose $(\pi_T, x_T) \ll (\pi''_T, x''_T)$ where $(\pi''_T, x''_T) \in V'$ and $x''_T > x_T$.

As $(\pi''_T, x_T) \ll (\overline{\pi}''_T, x_{\{m_P\}} = \widehat{m}_P)$, we need to ensure $(\overline{\pi}''_T, x_{\{m_P\}} = \widehat{m}_P) \notin V'$.

Say, $(\overline{\pi}''_T, x_{\{m_P\}} = \widehat{m}_P) \ll (\pi''_S, x''_S)$ where $x''_S < \widehat{m}_P$ and $(\pi''_S, x''_S) \in V'$. Then there is also (π'_S, x'_S) satisfying $x''_S < x'_S < m_P$ and $U_{m_P}(x'_S) > U_{m_P}(x''_T)$ which also blocks $(\overline{\pi}''_T, x_{\{m_P\}} = \widehat{m}_P)$.

Because $(\overline{\pi}''_T, x_{\{m_P\}} = \widehat{m}_P) \ll (\pi'_S, x'_S)$ together with $U_{m_P}(x'_S) > U_{m_P}(x''_T)$ implies $(\pi''_T, x''_T) \ll (\pi'_S, x'_S)$, we need to ensure $(\pi'_S, x'_S) \notin V'$. So we need to have $(\pi'''_T, x'''_T) \in V'$ such that $(\pi'_S, x'_S) \ll (\pi'''_T, x'''_T)$.

[39] In the case where there is no winner, the resulting lottery on π, $\ell(\pi)$, must satisfy $g(\ell(\pi)) < \widehat{m}_P$.

[40] $\ll$ is supported by a sequence where m_P joins S' to offer x'_S and m_{P+1} drops out to prevent a lottery on the proposals $(x_S, x'_S, \widehat{x}_{m_{P+1}})$.

Note that because all agents in $\overrightarrow{S}$ have exited, only agents with $i \geqslant m_P$ can change the outcome from x'_S. Moreover, as we claim that there is such (π'_S, x'_S), we can specify $(\pi'_S, x'_S; \mathbf{x}_{-S'})$ such that it shares the set of (unsuccessful) proposals in $[0, x''_S)$ with $(\pi''_S, x''_S; \mathbf{x}_{-S''})$. So say that a set $W'(x)$ of agents benefits from replacing x'_S with $x > x'_S$. Then the set of agents that benefits from replacing x''_S with x, $W''(x)$ satisifies $W'(x) \subseteq W''(x)$ and, hence, the set of feasible responses to x''_S, $\Omega(x''_S)$ satisfies, $\Omega(x'_S) \subseteq \Omega(x''_S)$. Loosely put, anything that blocks (π'_S, x'_S) also blocks (π''_S, x''_S).

So suppose that $(\pi'_S, x'_S) \ll (\pi'''_T, x'''_T)$ and $(\pi'''_T, x'''_T) \in V'$. Then also the relation $(\pi''_S, x''_S) \ll (\pi'''_T, x'''_T)$ holds, contradicting that $(\pi''_S, x''_S) \in V'$. Hence, $x''_T \notin V'$ and, as $(\pi_T, x_T) \in V'$, by internal stability, $(\pi_S, x_S) \notin V'$.

So cases (a) and (b) result in $(\pi_S, x_S) \notin V'$ and we can conclude that there is no (π_S, x_S) with $x_S < \widehat{m}_P$ in V' and, therefore, $\overline{\Pi} \subset V'$.

But elements in $\overline{\Pi}$ block all elements (π, x) with $x > \widehat{m}_P$ by the sequence $i \in \overleftarrow{S}$ drop out and $\{m_P\}$ offers $x_{\{m_P\}} = \widehat{m}_P$. Then there is no element in V' that blocks an element in $\underline{\Pi} \subset V'$. Finally, as $\pi_m = \{\{m_P\}\}$ is never blocked, $V' = V$.

6.10.2 The case $c > 0$

Stability of $V = \{(\pi_m, x_{\{m_P\}} = \widehat{m}_P)\}$

For c sufficiently small, the result that V indirectly dominates all positions (π, x_S) with $x_S \neq \widehat{m}_P$ follows from the argument established in 6.10.1: if the offer by the remaining coalition $T = S \backslash \overleftarrow{S}$ is $x_T \geqslant \widehat{m}_{P+1}$, m_P prefers to run as a singleton if $U_{m_P}(\widehat{m}_{P+1}) < U_{m_P}(\widehat{m}_P) - c$, i.e. if c is sufficiently small.

To show that there is no other element in V, suppose that $(\pi, x_S = \widehat{m}_P) \in V$ and observe that there can be no also-runs in this position.[41] Then $(\pi_m, x_{\{m_P\}} = \widehat{m}_P)$ indirectly dominates $(\pi, x_S = \widehat{m}_P)$ via the sequence: (1) S exits, $(\varnothing, x^0)$ is realized. (2) m_P enters for c sufficiently small and offers $(\breve{\pi}_m, \widehat{m}_P)$ which is winning. This argument shows that $V = \{(\pi_m, x_{\{m_P\}} = \widehat{m}_P)\}$ is externally stable and including another element results in a violation of internal stability. Hence, $V = \{(\pi_m, x_{\{m_P\}} = \widehat{m}_P)\}$ is stable.

6.11 Proof of lemma 4

We have to show that for $\pi \in \Pi \backslash \widetilde{\Pi}$ there exists $\pi' \in \widehat{\Pi}$ such that $\pi \ll \pi'$. The proof follows the proof of lemma 3: Let R_m be the constituency with m_ζ, the median of

[41] If $(\pi, x_S = \widehat{m}_P) \ll (\pi_m, x_{\{m_P\}} = \widehat{m}_P)$, the latter also indirectly dominates any partition with also-runs.

the median constituency. Let π be the status quo and assume that $g(\widehat{\varphi}(\pi)) > \widehat{x}_{m_\zeta}$. By the definition of m_P, $U_{m_\zeta}(\widehat{m}_P, 1) > U_{m_\zeta}(\widehat{\varphi}(\pi))$ and, by claim 3 in the proof of the theorem, for all constituency medians m_z with $\widehat{x}_{m_z} < \widehat{x}_{m_\zeta}$, $U_{m_z}(\widehat{m}_P, 1) > U_{m_z}(\widehat{\varphi}(\pi))$. Moreover, by assumption 1, $U_{m_P}(\widehat{m}_P, 1) > U_{m_P}(\widehat{\varphi}(\pi))$ and by lemma 2, for all i_z with $\widehat{x}_{i_z} < \widehat{m}_P$, $U_{i_z}(\widehat{m}_P, 1) > U_{i_z}(\widehat{\varphi}(\pi))$.

Construct π'' as follows: collect R_m and $\frac{Z-1}{2}$ other constituencies where the median voter's ideal point satisfies $\widehat{x}_{m_z} < \widehat{x}_{m_\zeta}$. In these constituencies, let all parties i_z with $\widehat{x}_{i_z} < \widehat{m}_P$ exit the race and let clones of m_P run in these constituencies. As m_P wins $\frac{Z+1}{2}$ constituencies, m_P wins a majority in parliament implements $(\widehat{m}_P, 1)$. Moreover, for ϵ sufficiently small, m_P enters for any status quo π with $\widehat{\varphi}(\pi) \neq (\widehat{m}_P, 1)$. So clearly, $\pi \ll \pi''$.

All parties running in π'', incur a cost $\epsilon > 0$ in the case of assumption 2 (or $z_k\epsilon > 0$ in the case of assumption 3). As only m_P is winning, all other parties $k \neq m_P$ are better off by exiting, resulting in $\pi' \in \widehat{\Pi}_m$ with $\pi \ll \pi'$.

6.12 Proof of proposition 6

The set of partitions where m_P runs in $\frac{Z+1}{2}$ constituencies is $\widehat{\Pi}_m \subset \widetilde{\Pi}_m$.

Consider part (a) of the proposition where $C \in \{0, \epsilon\}$: by lemma 4, $\widetilde{\Pi}_m$ is externally stable. Because all partitions in $\widetilde{\Pi}_m$ are pay-off equivalent, $\widetilde{\Pi}_m$ is internally stable, hence $\widetilde{\Pi}_m$ is stable.

Uniqueness: Let π_m^Z be the partition where m_P is the only party and runs in all Z constituencies. Suppose that there is $\pi'' \notin \widetilde{\Pi}_m$ such that $\pi_m^Z \ll \pi''$. Moreover, suppose that $U_i(\widehat{\varphi}(\pi'')) > U_i(\widehat{\varphi}(\pi_m^Z))$ for some i with $\widehat{x}_i > \widehat{m}_P$. By lemma 3, for all k with $\widehat{x}_k < \widehat{m}_P$, $U_k(\widehat{\varphi}(\pi_m^Z)) > U_k(\widehat{\varphi}(\pi''))$, so k do not enter and, hence, m_P must still be winning. Hence, $\widehat{\varphi}(\pi'') = (\widehat{x}_m, 1)$, contradicting that $\pi_m^Z \ll \pi''$. By external stability, $\pi_m^Z \in V$. Moreover, as π_m^Z blocks all partitions where m_P does not win, by internal stability, $V = \widetilde{\Pi}_m$ is the unique stable set.

Consider part (b) of the proposition where $C = \epsilon z_i$:

Note that for ϵ sufficiently small, partitions where only m_P runs in $z = 1, ..., \frac{Z-1}{2}$ constituencies do not dominate $\pi \in \widehat{\Pi}_m$ because they give the default pay-off $\ell^0 \neq (\widehat{m}_P, 1)$. Moreover, because of the increasing cost function, partitions where m_P runs in $z_m > \frac{Z+1}{2}$ constituencies are dominated by partitions in $\widehat{\Pi}_m$. By lemma 4, $\widehat{\Pi}_m$ is externally stable. Because all partitions in $\widehat{\Pi}_m$ are pay-off equivalent, $\widehat{\Pi}_m$ is internally stable, hence $\widehat{\Pi}_m$ is stable.

Uniqueness: Suppose there is $V' \neq V$. Let $\underline{\pi}_m^* \in \widehat{\Pi}_m$ be the partition where

m_P runs in all $\frac{Z+1}{2}$ constituencies with constituency median $\widehat{x}_{m_i} \leqslant \widehat{x}_{m_\zeta}$ and let $\overline{\pi}^*_m \in \widehat{\Pi}_m$ be the partition where m_P runs in all $\frac{Z+1}{2}$ constituencies with constituency median $\widehat{x}_{m_i} \geqslant \widehat{x}_{m_\zeta}$. Let $\Pi^{\widehat{z}}_m \subset \widetilde{\Pi}_m$ the set of partitions where m_P runs in $\widehat{z} \geqslant \frac{Z+1}{2}$ constituencies and let $\Pi^{\cup\widehat{z}}_m = \bigcup_{\widehat{z}=\frac{Z+1}{2}}^{Z-1} \Pi^{\widehat{z}}$.

Observations:

(ba) $\underline{\pi}^*_m$ and $\overline{\pi}^*_m$ together block every $\pi \notin \widehat{\Pi}_m$, so by external stability there must be $\pi' \in V'$ such that $\underline{\pi}^*_m \ll \pi'$.

(bb) There is no $\pi' \in \Pi^{\cup\widehat{z}}_m$ such that $\underline{\pi}^*_m \ll \pi'$.

(bc) π^Z_m where m_P runs in all Z communities is not blocked by any partition $\pi \in \Pi^{\cup\widehat{z}}_m$, so by external stability there is $\pi'' \in \Pi^{\cup\widehat{z}}_m$ such that $\pi'' \in V'$ and $\pi^Z_m \ll \pi''$.

Observation (ba) follows from lemma 4. Observation (bb) follows because m_P is indifferent between all partitions where she fields $z_m = \widehat{z}$ clones and her utility decreases in z_m for $z_m \geqslant \frac{Z+1}{2}$. To see that observation (bc) holds, suppose that there is $\pi \in \Pi \backslash \widetilde{\Pi}_m$ with $\pi^Z_m \ll \pi$. Then $g(\widehat{\varphi}(\pi)) \neq \widehat{m}_P$, so by lemma 2, $U_i(\widehat{\varphi}(\pi)) < U_i(\widehat{\varphi}(\pi^Z_m))$ for at least $\frac{Z+1}{2}$ constituency representatives so there is no deviation to change the outcome.

From observation (ba) and (bb) it follows that there must be $\pi' \in V'$ such that $\pi' \notin \widetilde{\Pi}_m$ and from observation (bc) it follows that there must be $\pi'' \in V'$ such that $\pi'' \in \Pi^{\cup\widehat{z}}_m$. But then $\pi' \ll \pi''$ via the following moves: m_P enters all Z districts, all agents $i \neq m_P$ exit and, finally, all clones of m_P which do not run in π'' exit. Hence, V' is not internally stable.

7 Appendix 2: Details

This part of the appendix provides additional details (not for publication).

7.1 Details of example 1

The payoff vector for $\{\{1,2,3\}\}$ is $(-.256-c, -.096-c, -.256-c)$, the payoff vector for $\{\{1,2\}\}$ is $(-.112-c, -.048-c, -.434)$ and the payoff vector for $\{\{1\},\{2\}\}$ is $(-.16-c, -c, -0.16)$.

7.2 Details of example 2

The payoff vector for $\{\{1\},\{2\}\}$ is $(-0.16, 0, -0.36)$, the payoff vector for $\{\{1\},\{2\},\{3\}\}$ is $(-0.364, -0.156, -0.444)$ and the payoff vector for $\{\{1\},\{2,3\}\}$ is $(-0.52, -0.154, -0.206)$.

The payoff vector for the variant of the example with $\widehat{x}_1 = 0.1$, $\widehat{x}_2 = 0.4$, $\widehat{x}_3 = 1$ and vector of votes for all candidates running $(0.3, 0.21, 0.49)$ is $\{\{1\},\{2\}\}$ is $(-0.09, 0, -0.36)$ and the payoff vector for $\{\{1\},\{2\},\{3\}\}$ is $(-0.416, -0.203, -0.319)$.

7.3 Example where a strong Nash voting equilibrium does not exist

In the following we construct an example for the case with pre-electoral coalition formation - as in scenario 1 - and rational voters where a voting equilibrium which is strong Nash does not exist.

Example 4. Assume voters are equally distributed on $[0,1]$ with $\widehat{x}_m = 0.5$ and candidates $N = \{1,2,3,4,5,6\}$ with $\widehat{x}_1 = 0.1$, $\widehat{x}_2 = 0.4$, $\widehat{x}_3 = 0.45$, $\widehat{x}_4 = 0.55$, $\widehat{x}_5 = 0.6$, $\widehat{x}_6 = 0.61$. Coalitions are $S_1 = \{1,4,5\}$ and $S_2 = \{2,3,6\}$.

If S_1 is predicted to win, voters have to vote their true preference on candidates in $\{1,4,5\}$ resulting in a payoff vector of $(-0.157, -0.052, -0.052, -0.067, -0.072, -0.073)$ and if S_2 is predicted to win, the vote results in a payoff vector of $(-0.173, -0.021, -0.013, -0.012, -0.019, -0.021)$

With S_1 winning, the median voter realizes $U_m = -0.057$ and with S_2 winning, the median voter realizes $U_m = -0.01$. So assume that S_2 is winning. But then, m and all voters with $\widehat{x} > 0.5$ can deviate and cast their vote for 4 and 5 instead. This results in $U_m = -0.009$ and a candidate payoff vector of $-0.233, -0.036, -0.02, -0.002, -0.000, -0.000$. However, as 4 and 5 are members of S_1, now S_1 is winning and, accordingly, voters in the range $[0, \frac{\widehat{x}_4 - \widehat{x}_1}{2})$ switch their vote to candidate 1 and voters in the range up to $\widehat{x}_m$ switch to candidate 4. But this results in the payoff vector for S_1.

So assume all voters with $\widehat{x} \geqslant \widehat{x}_2$ switch to vote for candidates in S_2. The resulting payoff vector for candidates is $(-0.289, -0.035, -0.022, -0.021, -0.0324, -0.036)$ and $U_m = -0.017$. So S_2 wins although votes need to adjust based on the assumption that S_2 wins. This completes the cycle. So from every configuration of voting strategies, there are agents who increase their utility by jointly deviating.

7.4 Details of the proof of proposition 4

The result is immediate if 2 gets a majority in the partition $\{\{1\},\{2\},\{3\}\}$. So assume instead that if all candidates run, any coalition of two candidates wins a majority of votes. For concreteness, assume that $\{\{2,3\}\} \succ_2 \{\{1,2\}\}$. The following relationships are straightforward for c sufficiently small:

$\{\{2,3\}\}$	$\succ_1, \succ_2, \prec_3$	$\{\{1\},\{2,3\}\}$	(a)
$\{\{2,3\}\}$	$\prec_1, \succ_2, \succ_3$	$\{\{1,2,3\}\}$	(b)
$\{\{1\},\{2\}\}$	$\prec_1, \succ_2, \succ_3$	$\{\{1,2\}\}$	(c)
$\{\{1,2\}\}$	$\succ_1, \succ_2, \prec_3$	$\{\{1,2,3\}\}$	(d)
$\{\{1,2\},\{3\}\}$	$\succ_1, \prec_2, \prec_3$	$\{\{1,2\}\}$	(e)
$\{\{1\},\{2\}\}$	$\prec_1, \succ_2, \succ_3$	$\{\{1,2\},\{3\}\}$	(f)
$\{\{2\},\{3\}\}$	$\succ_1, \succ_2, \prec_3$	$\{\{1\},\{2,3\}\}$	(g)

7.4.1 Show that for $c = 0$, $\{\{1\},\{2\}\}$, $\{\{2\},\{3\}\}$, $\{\{2\}\} \in LCS$ and for $0 < c < \epsilon$, $\{\{2\}\} \in LCS$.

Follows from proposition 3.

7.4.2 Show that $\{\{1,2\}\} \notin LCS$.

Assume 2 deviates to partition $\{\{1\},\{2\}\}$.

Suppose $\{\{2,3\}\} \in LCS$. The sequence $\{\{1\},\{2\}\} \xrightarrow{3} \{\{1\},\{2\},\{3\}\} \xrightarrow{\{2,3\}} \{1\},\{\{2,3\}\} \xrightarrow{1} \{\{\{2,3\}\}$ indirectly dominates the partition by (a) and (b) but, by our assumption on 2's preference, does not deter 2's deviation.

The sequence $\{\{1\},\{2\}\} \xrightarrow{3} \{\{1\},\{2\},\{3\}\} \xrightarrow{\{1,2\}} \{1,2\},\{\{3\}\} \xrightarrow{3} \{\{\{1,2\}\}$ does not indirectly dominate the partition because of 3's preference in (c), so 2's deviation is not deterred.

Finally, the sequence $\{\{1\},\{2\}\} \xrightarrow{1} \{\{2\}\}$ is indirectly dominating for $c > 0$ but does not deter 2's deviation. Hence, 2's deviation is not deterred and $\{\{1,2\}\} \notin LCS$.

7.4.3 Show that $\{\{2,3\}\} \notin LCS$.

Assume 2 deviates to partition $\{\{2\},\{3\}\}$.

The sequence $\{\{2\},\{3\}\} \xrightarrow{1} \{\{1\},\{2\},\{3\}\} \xrightarrow{1,2} \{\{1,2\},\{3\}\} \xrightarrow{3} \{\{1,2\}\}$ indirectly dominates the partition by (d) and (e) but, as shown in part 7.4.2, does not terminate in LCS. The alternative sequence starting with 1's move that terminates in $\{\{2\}\}$ does not indirectly dominate.

Finally, the sequence $\{\{2\},\{3\}\} \xrightarrow{3} \{\{2\}\}$ is indirectly dominating for $c > 0$ but does not deter 2's deviation. Hence, 2's deviation is not deterred and $\{\{2,3\}\} \notin LCS$.

7.4.4 Show that $\{\{1\},\{2,3\}\} \notin LCS$.

Assume 1 deviates to $\{\{2,3\}\}$

The sequence $\{\{2,3\}\} \xrightarrow{2} \{\{2\}\{3\}\}$ (and, for $c > 0$, $\xrightarrow{3} \{\{2\}\}$) is indirectly dominating and, by (f), does not deter 1's deviation.

7.4.5 Show that $\{\{1,2\},\{3\}\} \notin LCS$.

Assume that 3 deviates to $\{\{1,2\}\}$. The sequence $\{\{1,2\}\} \xrightarrow{2} \{\{1\}\{2\}\}$ (and, for $c > 0$, $\xrightarrow{1} \{\{2\}\}$) is indirectly dominating and, by (g), does not deter 3's deviation.

7.4.6 Show that $\{\{1,2,3\}\} \notin LCS$ and $\{\{1\},\{2\},\{3\}\} \notin LCS$

Assume that 2 deviates to $\{\{1,3\}\}$ (or $\{\{1\},\{3\}\}$). Suppose that 1 or 3 weakly prefers $\{\{1,3\}\}$ over an element in Π_m. Then by lemma 2 the other agent does not and is willing to drop out of the race in order to bring about a partition where 2 runs as a singleton and wins. The same result trivially applies in the case where neither 1 nor 2 weakly prefer $\{\{1,3\}\}$ over an element in Π_m.

7.4.7 For $c > 0$, show: $\{\{1\},\{2\}\}$ and $\{\{2\},\{3\}\} \notin LCS$.

As 1 (respectively, 3) exits, $\{\{2\}\} \in LCS$ obtains with the same utility from policy and lower cost for 1 (or, 3, respectively).

7.5 Details of example 2 (continued in section 4.3)

The result for $c = 0$ follows directly from proposition 6.10.

For the case $0 < c < \epsilon$, we have to show that V' is stable. Observe $(\pi_m, x_{\{m_P\}} = \widehat{m}_P) \ll (\pi', x_S = \widehat{m}_P)$ with $S = \{\{1,3\}\}$. In π_m, 2 realizes $U_2(\widehat{m}_P) - c$. Consider the sequence: 2 drops out of the race, all candidates realize $U(x^0) = -1 - c$, S enters and its members realize $U_1(\widehat{m}_P) - c$ and $U_3(\widehat{m}_P) - c$ while 2 realizes $U_2(\widehat{m}_P)$. So by internal stability, $(\pi_m, x_{\{2\}} = \widehat{m}_P) \notin V'$.

Show that $(\{\{1,2\}\}, x_{\{1,2\}} = \widehat{x}_P) \not\ll (\pi', x_S = \widehat{m}_P)$ for all $(\pi', x_S = \widehat{m}_P) \in V'$: if 1 exits, $(\{\{2\}\}, x_{\{2\}} = \widehat{x}_P)$ forms which is not stable and $(\{\{2,3\}\}, x_{\{2,3\}} = \widehat{x}_P)$ does not form. If 2 exits, $(\{\{1\}\}, x_{\{1\}} = \widehat{x}_1)$ becomes the status quo, from which $(\{\{1,2\}\}, x_{\{1,2\}} = \widehat{x}_P)$ may form but which leaves 2 in the same position as before the deviation.

The argument that $(\{\{1\},\{2\}\}, x_{\{2\}} = \widehat{x}_P) \not\ll (\pi', x_S = \widehat{m}_P)$ for all $(\pi', x_S = \widehat{m}_P) \in V'$ follows along the same line.

Similarly, we can show that $(\{\{2,3\}\}, x_{\{2,3\}} = \widehat{m}_P) \not\ll (\pi', x_S = \widehat{m}_P)$ for all $S \in V'$ and that $(\{\{2\},\{3\}\}, x_{\{2\}} = \widehat{m}_P) \not\ll (\pi', x_S = \widehat{m}_P)$ for all $(\pi', x_S = \widehat{m}_P) \in V'$.

Finally, show that $(\{\{1,3\}\}, x_{\{1,3\}} = \widehat{m}_P) \not\ll (\pi', x_S = \widehat{m}_P)$ for $S = \{1,2\}$, $\{2,3\}$ and $\{2\}$: if 1 exits, $(\{\{3\}\}, x_{\{3\}} = \widehat{x}_3)$ becomes the status quo. Although $(\{\{1,2\}\}, x_{\{1,2\}} = \widehat{x}_P)$ may form, it leaves 1 in the same position as before the deviation.

To complete the argument, no position where a policy other than $\widehat{x}_P$ is offered is stable: suppose that $\{\{2,3\}\}$ offers $x_{\{2,3\}} = x'$, $x' > \widehat{m}_P$. Now 1 can enter and form $\{\{1,2\}\}, x_{\{1,2\}} = \widehat{x}_P$.