Cycles and Optimistic Stability in Graphs: The Role of Competition, Veto Players and Moderators

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Abstract

This paper provides sufficient conditions for existence of a non-empty optimistic stable standard of behaviour (OSSB) in directed graphs. Graphs which are completely connected by single player arcs for each player always admit a (non-unique) OSSB. More specifically, a loop is broken by adding a single player move to some appropriately chosen default position. If there are nodes with a decision cycle over the move to the successor node, introducing veto players ensures that an OSSB exists which assigns a non-empty solution to every vertex along the equilibrium path.

Keywords: *Perfect foresight, intransitive choice, cyclicity in graphs, coalitions, theory of social situations, veto player.* ***JEL codes:*** *C70, C71, C72.*

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1 Introduction

The Theory of Social Situations (Greenberg, 1990) provides an alternative game-theoretic framework which is sufficiently rich to allow for the representation of cooperative and non-cooperative games. A "situation" is a complete description of the social environment. In our simplified setting the description of the environment consists of the set of players, feasible moves, alternatives and preferences over alternatives. Here, alternatives take the place of "positions" in Greenberg's terminology, and it is assumed that players - if they are decisive in such a move - can move the status quo from one alternative to another.[1] A graph is a natural representation of a situation which includes a game tree as a special case. Yet a situation can capture more complex environments which makes the Theory of Social Situations a promising framework for analysing environments where coalitions of players can move and their moves may result in cycles over a set of alternatives.

A standard of behaviour assigns to each alternative a "solution" or a set of moves which it recommends when this alternative is reached. A standard of behaviour is stable - in the sense of von Neumann and Morgenstern - if it is free of inner contradictions and it discredits all moves which it does not recommend. Greenberg has introduced Optimistic Stable Standard of Behavior ($OSSB$) as a solution concept for situations where farsighted players who contemplate a strategic move are optimistic about subsequent moves by other (optimistic) players when assessing possible outcomes resulting from their move. We derive conditions under which $OSSB$ exists and assigns a non-empty set of recommended moves to alternatives in environments which permit cycles.[2] We consider two types of cycles: "loops" and "decision cycles".[3] What is normally referred to as cyclicity in graphs is the presence of loops of the following form: In position x some coalition S_x can bring about y, in y some coalition S_y can bring about z and in z some coalition S_z can bring about x and every coalition strictly prefers the position it can bring about to the position in which it is decisive. Non-existence of $OSSB$ follows in the same way as non-existence of stable set, with which $OSSB$ is closely related, when the underlying domination relation is cyclical (von Neumann/Morgenstern, 1944).

[1] Our description of the environment follows Chwe (1994) and Xue (1998). Greenberg considers more general situations where each position which can be reached in the game consists of a set of players, a set of feasible outcomes and players' utility over these outcomes. Accordingly, moves between these positions described by an "inducement correspondence" are more generally defined as well.

[2] A corresponding solution concept for players who conservatively assess possible outcomes is Conservative Stable Standard of Behavior ($CSSB$). In our environment, $CSSB$ generally exists although it may assign the empty set as solution to some positions (see Xue, 1998, theorem 3.12).

[3] See Acemoglu/Egerov/Sonin (2012) for a discussion of the significance of the different concepts.

The second type of cycle we look at are "decision cycles" or the intransitivity of the decision where to move from some given position - represented by a node - as in the following example: In node v^0, coalition S_1 can bring about v_1, coalition S_2 can bring about v_2 and coalition S_3 can bring about v_3. Moreover, coalition S_i, $i = 2, 3$ prefers the node it can bring about to node v_{i-1} and S_1 prefers v_1 to v_3. No coalition wants to stay in v^0. In the Theory of Social Situations, the solution - or set of recommended moves - assigned to node v^0 is the empty set, i.e. the theory makes no predictions of where agents move from v^0.

While loops can be plausibly excluded in some situations, for example when time is seen as important in how the game unfolds (Pech, 2015), the intransitivity problem inflicts all situations where coalitions move. In this paper we show that adding sufficiently many single player moves to a graph resolves the problem of cyclicity and of intransitivity.

We establish our results in a directed graph with farsighted agents, using the framework of Xue (1998). A (generally non-unique) $OSSB$ exists if the graph is completely connected for each player by single player arcs, i.e. if each agent can move the status quo from any position to any other node. This result demonstrates that adding sufficiently many single-player moves to a graph resolves all problems of cyclicality.

We also provide specific rules of how to extend graphs to obtain $OSSB$ in situations which admit cycles: If we add to a loop one single player move by which an agent can move the status quo from the successor of her maximal node in the loop into some appropriately chosen default node, an $OSSB$ for this graph exists. If for any node where a decision cycle occurs there is one agent who can move the status quo from any successor node back to the original node, there exists an $OSSB$ which assigns a non-empty solution to nodes along an "equilibrium path" of the graph. In the former case the player whose move is added may be thought of as a moderator in a discussion (in a deliberative framework[4]) or as the chair of some legislative body (in a voting framework). In the latter case this player is essentially a veto player.

Section 2 defines the environment and solution concept. Section 3 derives an existence result for a graph which is completely connected for each player by single player moves. Section 4 provides rules on how to remedy specific problems of non-existence or emptiness of a solution by adding single player moves to a graph.

[4]See e.g. Piggins/Perote-Pena (2015).

2 Stability and farsightedness

Following Xue (1998), a social environment is represented by $\mathcal{G} = \left(N, Z, \{\succ_i\}_{i \in N}, \left\{\xrightarrow{S}\right\}_{S \subset N}\right)$. N is the set of individuals and Z is the finite set of alternatives or positions. $\{\succ_i\}_{i \in N}$ is the set of strict, not necessarily complete, individual preference relations on Z. We assume that $\succ_i$ is transitive. $\left\{\xrightarrow{S}\right\}_{S \subset N}$ is the set of effectiveness relations on Z and $a \xrightarrow{S} b$, $a, b \in Z$, means that S is decisive in a move from a to b and that when the status quo is a it may move the status quo to b. The directed graph $\phi(\mathcal{G})$ generated by $\mathcal{G}$ consists of the set of vertices (nodes) Z and a collection of arcs where $\langle a, b \rangle$ is an arc if $a \xrightarrow{S} b$ is an effectiveness relation. If $\langle a, b \rangle$ is an arc, b is said to be adjacent from a. A path α is a sequence of vertices $\langle v_1, v_2, \ldots v_k \rangle$ of length $k - 1$ where each v_{i+1} is adjacent from v_i. $\phi(\mathcal{G})$ might be cyclic, implying there are paths $\langle v_1, v_2, \ldots v_k \rangle$ with $v_k = v_1$. $\phi(\mathcal{G})$ is bounded, i.e. there exists a finite integer J such that each path α with $v_i \neq v_j$, $v_i \in \alpha$, $v_j \in \alpha$ has length less than J. We write Π for the set of paths admitted by $\left\{\xrightarrow{S}\right\}_{S \subset N}$ and Π_a for the set of paths originating in $a \in Z$. Preferences over paths are preferences over their terminal nodes, i.e. $\alpha \succ_i \beta$ if and only if $t(\alpha) \succ_i t(\beta)$ and $\alpha \succ_S \beta$ if $t(\alpha) \succ_i t(\beta)$ for all $i \in S$. [5] Our concept of stability is based on stable standard of behaviour:[6]

Definition 1. A standard of behaviour σ assigns for every $a \in Z$ a subset of Π_a, $\sigma(a) \subset \Pi_a$.

Definition 2. A standard of behaviour σ is optimistic internally stable if for all $a \in Z$, $\alpha \in \sigma(a)$ implies that there does not exist $b \in \alpha$, a coalition $S \subset N$ and $c \in Z$ such that $b \xrightarrow{S} c$ and there is some $\gamma \in \sigma(c)$ with $\gamma \succ_S \alpha$.

Definition 3. A standard of behaviour σ is optimistic externally stable if for all $a \in Z$, $\alpha \in \Pi_a \backslash \sigma(a)$ implies that there exists $b \in \alpha$, a coalition $S \subset N$, and $c \in Z$ such that $b \xrightarrow{S} c$ and for some $\gamma \in \sigma(c)$ it is true that $\gamma \succ_S \alpha$.

That is, an optimistic internally stable standard of behaviour σ consists of paths which are proof against optimistic deviations by coalitions. An optimistic deviation by coalition S is one which results in a node from where there is at least one (optimistic stable) path which leads to an outcome preferred by S. By external stability, σ accounts for all paths α which it excludes in that there is a

[5] Because preferences over paths are derived from preferences over nodes, only paths with a terminal node may be proposed to players: Even if a path follows a loop, it must terminate in a particular node.

[6] See Greenberg (1990), applied to graphs by Xue (1998).

vertex c which can be reached by some deviating coalition S from which originates at least one path γ in σ which dominates α for S. Finally:

Definition 4. A standard of behaviour σ is an $OSSB$ if it is optimistic internally and externally stable.

It is possible that at some node a different paths emanate which cannot be excluded by the external stability criterion and where different decisive coalitions want to move along different paths. In this case $OSSB$ makes no recommendation and assigns the empty set $\sigma(a) = \emptyset$. An $OSSB$ is said to exist for such a situation if the resulting assignment of paths (including the empty set at some nodes) satisfies the internal and external stability criterion. An $OSSB$ is called non-empty if it assigns a non-empty set $\sigma(a)$ to each $a \in Z$.

Xue (1998, lemma 3.6) shows that if σ is a stable standard of behaviour, there is at least one $a \in Z$ such that $a \in \sigma(a)$. This claim follows from external stability of σ: σ cannot be identically empty-valued. $a \in \sigma(a)$ must at least be satisfied for the terminal node of every path in $\sigma(a)$. Therefore, if a stable standard of behaviour exists, the set of stable positions, $E_\sigma = \{a \in Z | a \in \sigma(a)\}$, is non-empty.

3 Stability in graphs which are completely connected for each player

In order to see how farsightedness affects existence and uniqueness of equilibrium consider the following example:

Example 1. $Z = \{x, y\}$, the pay off vector for A and B at x is $(u^A(x), u^B(x)) = (1, 0)$ and at y is $(u^A(y), u^B(y)) = ((0, 1)$. The effectiveness relations are $x \xrightarrow{B} y$ and $y \xrightarrow{A} x$.

A position x is stable if $x \in \sigma(x)$. One can show that either x or y but not both are stable. First, suppose that x is stable, i.e. $x \in \sigma(x)$. Let ξ be a path originating in y and terminating in x and let θ be a path originating in x and terminating in y. By external stability of σ, $\xi \in \sigma(y)$. By internal stability of σ, $y \notin \sigma(y)$ and $\theta \notin \sigma(x)$. Similarly, stability can be constructed for $y \in \sigma'(y)$ and $\theta \in \sigma'(x)$. Therefore, the set of stable positions is $E_\sigma = \{x\}$ or $E_{\sigma'} = \{y\}$.

The following proposition generalizes our example to a graph with arbitrarily many players:

Proposition 1. *Assume that in a graph each agent can move the status quo from any given vertex to any other vertex. In that case, a non-empty OSSB non-uniquely exists. Every node which is maximal for at least one agent can be supported as an OSSB unless it is Pareto-dominated.*

Proof. See part 1 of the appendix. □

Sacrificing the result on Pareto-undominated nodes, the conditions of proposition 1 can be slightly weakened:

Corollary 1. *Assume that in a graph each agent can move the status quo from any given vertex to a point in her maximal set. A non-empty $OSSB$ for this graph non-uniquely exists.*

Proof. Follows from the proof of proposition 1. □

Yet even a completely connected graph where the moves of individual players are further restricted does not in general admit an $OSSB$.[7] Note that including moves by coalitions of agents does not change the result of the proposition: In a setting where what a coalition can do coincides with what any of its members can do individually, admitting moves by coalitions does not change the set of outcomes.

4 Specific rules for graphs with cycles

The assumption that players can move the outcome anywhere they want - or at least to points in their maximal set - is too restrictive to make the result of proposition 1 particularly useful for the analysis of institutions. Yet the crucial idea of proposition 1 - that adding sufficiently many single player effectiveness relations can resolve a situation which gives rise to cycles - can be applied to special cases where more useful policy rules can be established. In this section we show that we can add single player effectiveness relations to graphs with loops or decision cycles such that existence of a non-empty $OSSB$ can be ensured.

In the case of a decision cycle, adding a "veto" player resolves the intransitivity problem and allows us to assign at least one non-empty $OSSB$.

Example 2. In a congressional bargaining situation with three legislators, $\{A, B, C\}$, a coalition of two can vote a spending bill into law unless the president uses her veto to block the bill from becoming law. Player D is the president and the presidential veto is interpreted as moving the bill back to the floor of the legislative assembly for consideration. This game is depicted in figure 1 with $\mathbf{u} = (u_A, u_B, u_C, u_D)$ representing the vector of pay offs for any node. The status quo has a pay off of zero.

[7] Modify the graph of example 3 below such that it is completely connected with single player moves where each agent can move into the node which gives her the second highest pay off. In this variant of example 3, suppose $\sigma(v_2) = v_2$, hence γ must be blocked and thus $v_3 \in \sigma(v_3)$, contradicting internal stability (C would want to move away from v_3 to receive her second highest pay off in v_2).

In the absence of a presidential veto this game clearly produces a decision cycle. Trivially, all terminal nodes are stable and $OSSB$ assigns the empty set to node v^0, i.e. $\sigma(v^0) = \emptyset$. Now add the arcs representing the presidential veto. The $OSSB$ which we constructed for the graph without the veto still satisfies the conditions for stability: As the president perceives the situation on the floor to be chaotic, she may not use her veto no matter which decision has been taken. Formally, the president does not deviate from any terminal node to v^0 when $\sigma(v^0) = \emptyset$, so the terminal nodes are stable. However, there is now an alternative $OSSB$ where the president vetoes all bills unless they take her to node v_1 where she realizes a pay off of 2. Note that this $OSSB$ is particularly appealing: In the first $OSSB$ the president would not even veto a bill which gives her the lowest pay off which she can attain in the game.

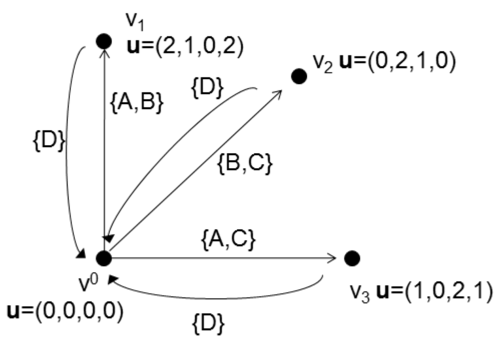

Figure 1: A decision cycle (example 2)

Proposition 2. *Assume that a graph contains no loops and that at any vertex v_i where a decision cycle occurs, there is an agent i^* with strict order $\{\succ\}_{i^*}$ on Π_{v_i} who can move back into "default" position v_i from any node which is adjacent to v_i. Then an $OSSB$ for the graph exists which assigns a non-empty solution along an equilibrium path.*

Proof. See part 2 of the appendix.

□

An $OSSB$ may fail to exist if the graph contains a loop. This is demonstrated in the following example:

Example 3. The set of agents is $\{A, B, C\}$ and the set of vertices $Z = \{v_1, v_2, v_3\}$ with outcomes $\mathbf{u} = (u^A, u^B, u^C)$ given as $\mathbf{u}(v_1) = (1, 0, 2)$, $\mathbf{u}(v_2) = (0, 2, 1)$, $\mathbf{u}(v_3) = (2, 1, 0)$ and the effectivity relations: $v_1 \overset{\{A,B\}}{\longrightarrow} v_3$ on arc α, $v_3 \overset{\{B,C\}}{\longrightarrow} v_2$ on arc β, $v_2 \overset{\{A,C\}}{\longrightarrow} v_1$ on arc γ.

The graph is depicted in figure 2. To show that an $OSSB$ for this graph does not exist, say for example $v_3 \in \sigma(v_3)$. By internal stability, $v_2 \notin \sigma(v_2)$, hence $\gamma \in \sigma(v_2)$ and $v_1 \in \sigma(v_1)$. However, the latter violates internal stability.

Adding one player who can move from any node in the loop back to some appropriate "default" position in the loop solves this problem.[8] To see this, consider the following modification of example 3:

Example 4. Assume effectiveness relations are as in example 3 and add the following effectiveness relation $v_2 \overset{A}{\longrightarrow} v_1$ for A which we denominate arc γ^*.

We can specify $v_3 = \sigma(v_3)$ together with $\gamma^* \in \sigma(v_2)$ and $\alpha \in \sigma(v_1)$ to obtain the unique $OSSB$. A manages to realize her maximal pay off because there are agents who want to move from her default node v_1 to node v_3. In any cycle this must be true at least of the successor of the default node. Note that unlike coalition $\{A, C\}$ which is effective for γ in the original graph, A would want to "farsightedly" deviate from v_2, expecting to get v_3.

We show that for any loop, if there is one agent who can move the status quo from any position in the cycle to some appropriate "default" position an $OSSB$ always exists. We refer to a cycle as follows:

[8] Converting the graph into a completely connected graph does not suffice as can be seen by adding the following effectivity relations: $-\beta$: $v_2 \overset{\{A,B\}}{\longrightarrow} v_3$, $-\gamma$: $v_2 \overset{\{B,C\}}{\longrightarrow} v_1$, $-\alpha$: $v_1 \overset{\{A,C\}}{\longrightarrow} v_3$ where we refer to the reverse of an arc by denoting it with a "$-$"-sign. Note that now each coalition of two players can move from any vertex of the game to a vertex where its players realize a positive pay off. Yet an $OSSB$ for the resulting graph fails to exist.

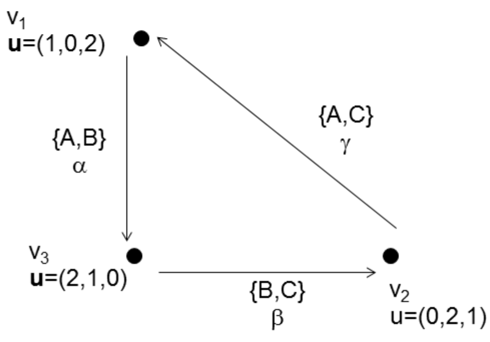

Figure 2: A loop (example 3)

Definition 5. A graph is a cycle C if it contains only nodes $\{v_1, ..., v_n\}$ such that for any two successive nodes, v_t, v_{t+1}: $v_t \xrightarrow{S_t} v_{t+1}$, $v_{t+1} \succ_{S_t} v_t$ and $v_{n+1} = v_1$.

The indirect dominance relation identifies paths which are potentially attractive for successive effective coalitions of agents because the terminal point of the path improves upon the status quo which they find.

Definition 6. v_b indirectly dominates v_a, or $v_a \ll v_b$, if there exists $v_1, v_2, ..., v_n$ and $v_i \xrightarrow{S_i} v_{i+1}$ with $v_1 = v_a$ and $v_n = v_b$ such that $v_b \succ_{S_i} v_i$.

Let $\Xi(v_i) = \{\psi \in \Pi(v_i) | v_i \ll t(\psi)\}$ be the set of paths which indirectly dominate default position v_i. Note that if $v_i \notin \sigma(v_i)$, then by external stability, there must be $v^* \in \Xi(v_i)$ which blocks v_i.

Proposition 3. *Take a cycle C and choose v^0 and i^* such that there is a strict maximum for i^*, v^* with $\langle v^0, v^* \rangle \in \Xi(v^0)$. Let v_a be the direct successor of v^*. Obtain C' by adding an individual effectiveness relation $v_a \xrightarrow{i^*} v^0$. Then an OSSB for C' exists.*

Proof. See part 3 of the appendix. □

To see why proposition 3 holds it is useful to think about why stability may fail to hold. The following discussion is illustrated in figure 3. Stability requires that at least one node v^* in a cycle satisfies $v^* \in \sigma(v^*)$. By external stability, the successor node of v^*, v_a needs to be blocked and the blocking node needs to be stable itself. Say the blocking node is v_b with $v_b \in \sigma(v_b)$. In the simplest case, we obtain a contradiction because v^* blocks v_b - i.e. because $\langle v_b, v^* \rangle \in \Xi(v_b)$ - thus violating internal stability. But this "moment of truth" may be deferred if v_b has a successor which needs to be blocked but the blocking node is blocked by v^* and so on.[9] If, however, i^* can move from v_a back to some default node which further leads to v^*, she can ensure that v^* itself blocks v_a without encountering a contradiction. Interestingly, it is sufficient that the cycle is broken once by blocking the successor of v^* and an $OSSB$ can be found for the remainder of the cycle. Hence, the crucial link for player i^* is $\langle v_a, v^0 \rangle$ and adding this link alone to C ensures existence of an $OSSB$.

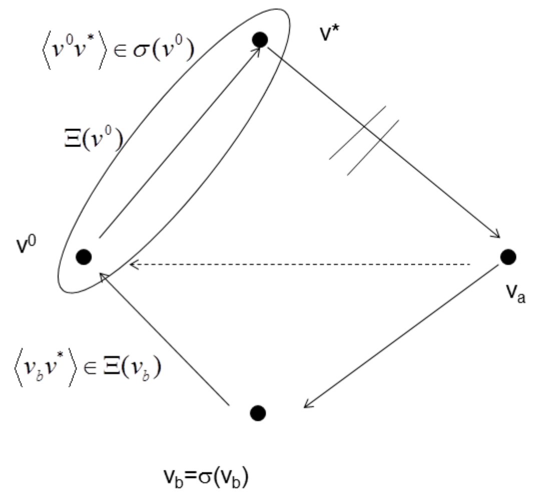

Figure 3: Illustrating the intuition of proposition 3.

The following example illustrates that the condition of proposition 3 that v^* is a maximum cannot be dispensed with:

[9]Part 3 of the appendix contains a characterization of $OSSB$ on a cycle.

Example 5. Consider the same effectiveness relations as in example 3 and add the effectiveness relations for A: $v_3 \xrightarrow{A} v_2$, corresponding to arc β^*, $v_1 \xrightarrow{A} v_2$, corresponding to arc $-\gamma^*$.

Now neither the default node v_2 nor the node $v_1 \in \Xi(v_2)$ is a maximum. An $OSSB$ for the modified graph does not exist.

5 Conclusion

Existence is an important feature even if the solution is not unique. For example, Shitovitz (1994) shows that there non-uniquely exists an $OSSB$ for extensive form finite action games of perfect information with no chance moves which can be represented as a game tree. The largest consistent set (Chwe, 1994) is unique yet it may contain many outcomes. Although it is sometimes possible to characterize these outcomes in terms of common desirable properties (such as in Pech, 2012), the theory makes no prediction of which outcome eventually prevails. In all these cases multiplicity leaves us with the possibility that the most likely or convincing solution can be established outside of the framework.

Although stability does not necessarily give an answer to the question of which outcome is ultimately going to prevail it shows which outcomes may potentially prevail. In this sense the paper offers a new perspective on Gordon Tullock's question "Why so much stability?". Its striking proposition is that sufficiently many effectiveness relations in a graph ensure that a stable solution exists. This contrasts to approaches which impose a solution by excluding effectiveness relations through the institutional framework such as Shepsle/Weingast (1981). In our framework effectiveness relations or rules of the game may empower a particular player or, if the rules of the game are competitive, players may settle for one particular outcome.

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6 Appendix

6.1 Proof of proposition 1

Let M_i be the set of maximal elements in Z with respect to $\succsim_i$. Initially, assume that M_i is a singleton and let $M_i = \{x\}$. For all $y \in Z$ let $\sigma(y) = \{\alpha \in \Pi_y | t(\alpha) = x\}$. Then σ is stable: Internal stability follows because x is maximal for i. External stability follows from the assumption that any player and in particular i can move from any y to x. Furthermore, σ assigns a singleton: let $\gamma \in \Pi_y$ with $t(\gamma) \neq x$ and suppose $\gamma \in \sigma(y)$ and $\alpha \in \sigma(y)$. Clearly, σ is not internally stable.

Now assume that M_i contains more than one element and that $x, x' \in M_i$. Initially, suppose neither Pareto-dominates the other (i.e. there is k with $x \succ_k x'$ and there is j with $x' \succ_j x$). Now either $\sigma(y) = \{\alpha \in \Pi_y | t(\alpha) = x\}$ or $\sigma(y) = \{\beta \in \Pi_y | t(\beta) = x'\}$ is internally and externally stable: Let $\gamma = \langle x', x \rangle$ and say $\gamma \in \sigma(x')$ all $x' \neq x$. In that case, by external stability, $\beta \notin \sigma(y)$ and by internal stability, $x \in \sigma(x)$. Therefore, $E_\sigma = \{x\}$ is a set of stable positions, similarly, $E_{\sigma'} = \{x'\}$ is a set of stable positions.

Next assume that x' is Pareto-dominated, i.e. there is k with $x \succ_k x'$ and but there is no j with $x' \succ_j x$. x' is not a stable outcome: Suppose that $x' \in \sigma(x')$. By external stability, $x \in \sigma(x)$. But then k wants to deviate to x, and because she can move to any node, this is feasible. Hence, internal stability is violated.

Finally, consider the case where every player in N is indifferent between x and x'. In that case, $\sigma(y) = \{\alpha \in \Pi_y | t(\alpha) = x\} \cup \{\beta \in \Pi_y | t(\beta) = x'\}$ is internally and externally stable. Therefore, $E = \{x, x'\}$ is a set of stable positions. In all cases non-emptiness of σ is guaranteed because the stable positions block all other positions.

6.2 Proof of proposition 2

For a node $v_t \in \psi$ define $\psi_{|v_t}$ as the truncation of ψ starting in v_t and terminating in $t(\psi)$. Let $\psi \in \Pi_{v^0}$ be a path such that $\psi_{|v_t} \in \sigma(v_t)$ for all $v_t \in \psi$, $t > 0$,[10] and let Ψ denote the set of all such paths.

Consider the set of successors of v^0, $P^0 = \{a_i | a_i \text{ is adjacent from } v^0\}$. If for all $\sigma(a_i) = \emptyset$ then $\sigma(v^0) = v^0$.

So say there is exactly one $a_i \in P^0$ with $\gamma \in \sigma(a_i)$ and $\sigma(a_j) = \emptyset$ for $a_j \in P^0$, $j \neq i$. Because a_i is adjacent to v^0, there is T_i which effects a_i, i.e. $v^0 \xrightarrow{T_i} a_i$. Now, either $\gamma \succ_{T_i} v^0$ holds and $\langle v^0 a_i \rangle \cup \gamma \in \sigma(v^0)$ or $\gamma \not\succ_{T_i} v^0$ and $v^0 \in \sigma(v^0)$. Next, say that there is a_1 with $\gamma_1 \in \sigma(a_1)$ and $v^0 \xrightarrow{T_1} a_1$ and a_2 with $\gamma_2 \in \sigma(a_2)$ and $v^0 \xrightarrow{T_2} a_2$. Furthermore, suppose that $\gamma_1 \succ_{T_1} \gamma_2$ and $\gamma_2 \succ_{T_2} \gamma_1$ and that γ_1 and $\gamma_2 \in \Psi$, i.e. T_1 and T_2 want to move away from v^0. These conditions are sufficient for $\sigma(v^0) = \emptyset$ and there is one $OSSB$ where player i^* does not want to move back to v^0 because this creates chaos.

However, there is another $OSSB$: Assume γ_1 is the path preferred by i^*. Then $\langle v^0 a_1 \rangle \cup \gamma_1 \in \sigma(v^0)$ and $\gamma_2 \notin \sigma(a_2)$ because i^* wants to move back from a_2 to v_0. Note that we may have $\sigma(a_2) = \emptyset$.[11] The argument can be extended to any finite number of paths originating in P^0.

We still have to show that $\sigma(a_1) \in \Psi$ as assumed. The proof is by induction:

Consider the direct predecssor v_{T-1} of terminal node v_T. By construction, a terminal node v_T satisfies $v_T \in \sigma(v_T)$.

As demonstrated for v^0, there is an $OSSB$ with $\sigma(v_{T-1}) \neq \emptyset$. Let this path be $\gamma | v_{T-1}$ which is maximal for i^{T-1} in P^{T-1}. Now consider the predecessor node v_{T-2}. For each node $b \in P^{T-2}$ we similarly construct $\sigma(b) \neq \emptyset$ and an $OSSB$ with $\sigma(v_{T-2})$. Induction over all $T - s$, $s = 1, ..., T$ shows that an $OSSB$ exists where along an equilibrium path ψ, $\sigma(v_t) \neq \emptyset$, $t = 0, ..., T$.

6.3 Proof of proposition 3

The following statement generalizes the illustration in the text:

[10]Following Xue, 1998, lemma 3.5, any stable path satisfies this "truncation property".

[11]This can be avoided if i^* is also introduced as veto-player at node a_2.

Remark. Generally, an $OSSB$ can be constructed iff there exists a sequence $v^{(1)}, ..., v^{(k)}, ..., v^{(m)}$ such that for $v^{(k)}$, its immediate successor $v_1^{(k)}$ and for their successor in the (cyclical) sequence, $v^{(k+1)} (\mathrm{mod}\, m)$ we have $\left\langle v_1^{(k)}, v^{(k+1)} \right\rangle \in \Xi(v_1^{(k)})$ and $\left\langle v^{(k)}, v^{(k+1)} \right\rangle \notin \Xi(v^{(k)})$.

Proof. The set of stable positions $\{v^{(1)}, ..., v^{(k)}, ..., v^{(m)}\}$ with $v^{(k)} \in \sigma(v^{(k)})$ corresponds to an $OSSB$: By the first statement, paths terminating in the stable verteces block all other verteces, so σ is externally stable. By the second statement, there is no path terminating in a stable vertex which blocks another stable vertex, so σ is internally stable. Moreover, a path α blocks a vertex v_i only if $\alpha \in \Xi(v_i)$. This shows equivalence between the statements in the remark and definition 1-4 establishing an $OSSB$ when applied to a cycle C. □

In a cycle, a sequence satisfying external stability can always be constructed because there is always an effective coalition which wants to move to the next node and no vertex $v_i \in C$ satisfies $\langle v_i, v_i \rangle \in \Xi(v_i)$. Note that as a consequence, in any $OSSB$ $v_i = \sigma(v_i)$ must hold at least at one node. An $OSSB$ does not exist if every externally stable sequence results in $\left\langle v^{(m)}, v^{(1)} \right\rangle \in \Xi(v^{(m)})$.

To prove the proposition, take C, choose the strict maximum for i^* on C, v^1, and add the individual effectiveness relation $v_1^{(1)} \xrightarrow{i^*} v^0$ for the direct successor of $v^{(1)}$, $v_1^{(1)}$, and for some node $v^0 \in C$ such that $\left\langle v^0, v^{(1)} \right\rangle \in \Xi(v^0)$.

In the graph C', $v^{(1)} \in \sigma(v^{(1)})$ irrespective of the remainder of the sequence establishing external stability:

Because $v^{(1)}$ is maximal for i^*, i^* wants to deviate from any path passing through $v_1^{(1)}$, i.e. $\left\langle v_1^{(1)}, v^{(1))} \right\rangle \in \sigma(v_1^{(1)})$. Hence, the remaining problem is to construct an $OSSB$ for $C_{v_2^{(1)}}^{v^{(1)}}$ which is a path starting in the successor of $v_1^{(1)}$ and with terminal point $v^{(1)}$. This path is acyclic, so an $OSSB$ for this graph exists (see Xue, 1998, proposition 3.9).