A Median Voter Theorem for Proportional Representation Systems with Firm Preelectoral Coalitions*

Gerald Pech†

March 15, 2010

Abstract

We define the stable set of a party formation game with farsighted citizen candidates under proportional representation. Voters are sincere, parties are identified with candidate positions and can enter firm preelectoral coalitions. A coalition government is associated with a lottery over the ideal positions of its constituent parties. If preferences are strictly concave, in every outcome in the unique farsighted stable set the candidate representing the median voter wins the election as a singleton.

Keywords: Median Voter, Proportional Representation, Endogenous Political Parties, Farsighted Stable Set.

JEL codes: C72, D72.

1 Introduction

Median voter theorems, following Downs (1957), have been formulated for the case of two party spatial competition under plurality rule (Calvert 1985). This paper establishes a median voter theorem for proportional representation systems in a citizen candidate model (Besley/Coate, 1997) with sincere

*For helpful comments I am grateful to Emiliya Lazarova, Nadeem Naqvi and Ashley Piggins.

†American University in Bulgaria, 1, Izmirliev Square, 2700 Blagoevgrad, Bulgaria, e-mail: gp@geraldpech.net.

voters where parties can agree to form firm preelectoral coalitions. We consider moves by policy-motivated agents which are farsighted and optimistic. Such agents are willing to engage in a deviation when they see the possibility for them to be better off as a result, taking into account subsequent deviations by other agents. The solution concept we apply is the stable set based on an indirect - hence farsighted - dominance relation (Harsanyi, 1974, Chwe, 1994). When deriving the stable set, it is outcomes within the set, hence stable outcomes, which are considered as objections against other outcomes. In every stable outcome, the candidate representing the median voter runs as a singleton and wins the election.

Citizen candidate models of proportional representation have been proposed by Hamlin/Hjortland (2000) and Bandyopadhyay/Oak (2004).[1] The latter consider government formation and obtain the median voter outcome as one of possibly many Nash equilibria. Cox (1990) compares Nash equilibria with office seeking politicians under plurality rule and proportional representation. In either case there are divergent equilibria (if they exist) when there are more than two candidates.Taking into account that the median voter outcome is also the core of a voting game in a committee[2] it may not be entirely surprising to obtain stability of the median voter outcome even in an institutionally quite different setting. The fact that the farsighted stable set is unique is, however, quite remarkable.[3]

2 Definitions

A community N is made up of an odd and finite number of citizens. Citizen i has a strictly concave utility function $u^i(x)$ on a policy variable $x \in [0,1]$ with ideal point $\widehat{x}_i \in [0,1]$ and $u_i(\widehat{x}_i) = 0$. Citizens can be consecutively ranked by their ideal point. At the party formation stage, they decide to run for election or drop out of the race. A citizen who runs is identified as a party. Parties may form a firm preelectoral coalition before a vote is held. Let $\pi = \{S_1, .., S_K, D\}$ be a partition of N into coalitions S_k of parties - possibly singletons - and the coalition D of citizens which do not run. Π is the set of all partitions.

[1] See Pech (2009) on the effect of party structure in an elected assembly.

[2] Anesi (2006) relates Neumann-Morgenstern stable set to the core of a voting game with farsighted agents and an equilibrium process.

[3] Shino (2009) shows non uniqueness in Hotelling's location game.

Given π, at the voting stage a citizen votes sincerely for the party whose ideal point gives her the highest utility realization. When indifferent, the voter tosses a fair coin. We assign government the coalition S with an absolute majority. If no coalition has an absolute majority, the grand coalition $N \backslash D$ forms the government.[4] If no party runs, the default policy $x_0 = 0$ is realized. Each government party is recognized to select the policy with a probability equal to her vote share relative to all parties in government and, when recognized, selects its ideal policy point. The set of ideal points in S and the vector of party weights $\mathbf{p}$ determine the policy lottery $\ell(S, \mathbf{p})$ associated with π. Agent i evaluates $\ell(S, \mathbf{p})$ with her von Neumann-Morgenstern utility $U^i(\ell) = (u^i(\widehat{x}_j))'_{j \in S}\mathbf{p}$. For coalition T, $\pi' \succ_T \pi$ if $U^i(\ell(\pi')) > U^i(\ell(\pi)) \forall i \in T$.

What a coalition T can do at the party formation stage at some status quo partition π is represented by the effectiveness relation $\pi \xrightarrow{T} \pi'$ (Chwe, 1994). In moving from π to π' members of T may form a preelectoral coalition $S \subseteq T$, drop out of the race by joining D, or leave any coalition of which they are member. If T moves to π', π' becomes the new status quo at which another coalition might move. N, Π, U and the set of effectiveness relations $\{\xrightarrow{T}\}$ define a game $\Gamma = (N, \Pi, \{\succ_i\}_{i \in N}, \{\xrightarrow{T}\}_{T \subseteq N, T \neq \emptyset})$.

Agents are farsighted: When contemplating a deviation they anticipate incentives of subsequent coalitions to deviate. A partition π' may be reached from partition π if it indirectly dominates π, that is if every deviating coalition T_k in a sequence leading from π to π' prefers π' to the partition π_k from which it deviates.

Definition 1 *π is indirectly dominated by π', or $\pi \ll \pi'$, if there exists a sequence $\pi_k \xrightarrow{T_k} \pi_{k+1}$, $k = 0, .., K-1$ with $\pi_0 = \pi$ and $\pi_K = \pi'$ such that $\pi' \succ_{T_k} \pi_k$ for all T_k.*

Agents are optimistic: They are willing to deviate from a partition π if at least one of the partitions which may be reached after a deviation leaves this coalition better off. In order to define stable outcomes for farsighted optimistic agents we define the farsighted stable set for a game $(\Pi, \ll)$ (Chwe, 1994):

Definition 2 *V is a stable set for $(\Pi, \ll)$ if (1) V is farsighted internally stable: For all π, $\pi' \in V$, neither $\pi \ll \pi'$ nor $\pi' \ll \pi$, and (2) V is farsighted externally stable: $\pi \in \Pi \backslash V$ implies that there is $\pi' \in V$ and $\pi \ll \pi'$.*

[4] Essentially, bargaining takes place in parliament rather than cabinet.

The farsighted stable set is internally stable, i.e. free of contradictions: No farsighted optimistic coalition wants to deviate to another partition in V. It is also externally stable, i.e. it accounts of all partitions which it excludes: A partition is not in V only if a farsighted optimistic coalition wants to deviate from it. The following example illustrates this environment:

Example 1 *Suppose that $N = \{1, 2, 3\}$ with $\widehat{x}_1 = 1/3$, $\widehat{x}_2 = 1/2$, $\widehat{x}_3 = 2/3$ and $u^i = -|x - \widehat{x}_i|^2$.*

If all three run as singletons, 2 prefers forming a coalition with either 1 or 3. Once the coalition has formed, the remaining agent wants to drop out of the race, giving 2 an incentive to leave the coalition and run as a singleton in which case she wins the election. Thus, in a farsighted stable equilibrium, 2 runs and wins and either 1 or 3 may also run.

3 Results

Our first result characterizes lotteries $\ell(S, \mathbf{p})$ induced by a coalition S in relation to $(\widehat{x}_m, 1)$, the ideal policy of median voter m with probability 1. Lemma 1 (a) relates the expected value of ℓ, $g(\ell)$, to $\widehat{x}_m$. By lemma 1 (b), if one agent weakly prefers ℓ, agents on the opposite side of the median strictly prefer $(\widehat{x}_m, 1)$. By lemma 1 (c), at least one agent on the boundary of S and all more extreme outsiders on her side of S strictly prefer $(\widehat{x}_m, 1)$ to ℓ.

Lemma 1 *(a) Let $\ell(S, \mathbf{p})$ be the policy lottery associated with a coalition $S \neq \{m\}$. If an agent with $\widehat{x}_i > \widehat{x}_m$ weakly prefers ℓ to $(\widehat{x}_m, 1)$, the centre of gravity of ℓ, $g(\ell) > \widehat{x}_m$. (b) If for one agent with $\widehat{x}_i > \widehat{x}_m$, $\ell(S, \mathbf{p}) \succsim_i (\widehat{x}_m, 1)$ is true, then for all agents j with $\widehat{x}_j < \widehat{x}_m$, $(\widehat{x}_m, 1) \succ_j \ell(S, \mathbf{p})$ holds. (c) Let s_1 the left-most and s_J the right-most member of S. Then all agents with $\widehat{x}_i \leq \min(\widehat{x}_m, \widehat{x}_{s_1})$ or all agents with $\widehat{x}_i \geq \max(\widehat{x}_m, \widehat{x}_{s_J})$ strictly prefer the policy point of the median voter to $\ell(S, \mathbf{p})$.*

Proof. (a) Suppose $\widehat{x}_i > \widehat{x}_m$ and $U^i(\ell) \geq u^i(\widehat{x}_m)$. Construct ℓ' on $[0, \widehat{x}_i]$ and ℓ'' on $(\widehat{x}_i, 1]$ such that $\ell = p\ell' + (1-p)\ell''$. Note that $g(\ell) = pg(\ell') + (1-p)g(\ell'')$.

Case 1) $U^i(\ell') \geq u^i(\widehat{x}_m)$. Because ℓ' is defined on $[0, \widehat{x}_i]$, $g(\ell') > \widehat{x}_m$ from Jensen's inequality. Because $g(\ell'') > \widehat{x}_m$, by construction $g(\ell) > \widehat{x}_m$.

Case 2) $U^i(\ell') < u^i(\widehat{x}_m)$. To get a lower bound on $g(\ell)$ with $g(\ell'') \geq \widehat{x}_i$, we replace ℓ'' by $(\widehat{x}_i, 1)$ and note $pU^i(\ell') + (1-p)u^i(\widehat{x}_i) \geq U^i(\ell) \geq u^i(\widehat{x}_m)$.

Using $u^i(\widehat{x}_i) = 0$ it follows that $pU^i(\ell') \geq u^i(\widehat{x}_m)$. By Jensen's inequality, $pu^i(g(\ell')) > pU^i(\ell') \geq u^i(\widehat{x}_m)$. For $g(\ell') > \widehat{x}_m$, $g(\ell) > \widehat{x}_m$ follows. Hence, assume $g(\ell') \leq \widehat{x}_m$. Define a concave function $-f(\widehat{x}_i - x) = u^i(x)$ on $x \leq \widehat{x}_i$. $-pf(\widehat{x}_i - g(\ell')) > -f(\widehat{x}_i - \widehat{x}_m)$ only if $p(\widehat{x}_i - g(\ell')) < \widehat{x}_i - \widehat{x}_m$. Using the definition of $g(\ell)$ with $g(\ell'') \geq \widehat{x}_i$, immediately gives $g(\ell) > \widehat{x}_m$.

(b) Follows immediately from (a).

(c) The claim is trivial if S is a coalition of agents on one side of the median possibly including m (i.e. min or max operator binds). Consider $\widehat{x}_{s_1} < \widehat{x}_m < \widehat{x}_{s_J}$. For s_1 and s_J, ℓ is a one-sided lottery. By Jensen's inequality, $|g(\ell) - \widehat{x}_i| \geq |\widehat{x}_m - \widehat{x}_i| \Rightarrow U^i(\ell) < u^i(\widehat{x}_m)$. But $g(\ell)$ cannot simultaneously be closer to $\widehat{x}_{s_1}$ and $\widehat{x}_{s_J}$ than $\widehat{x}_m$. ∎

Lemma 2 extends lemma 1 (b) by comparing ℓ to a lottery ℓ^* which is induced by $S^* \in \pi^*$ with only m and its next neighbor $m-1$ running and forming the coalition S^*.

Lemma 2 *Given π^*, there are no two agents d and h with $\widehat{x}^d \leq \widehat{x}_{m-1}$ and $\widehat{x}^h \geq \widehat{x}_m$ and a lottery ℓ such that $\ell \succ_d \ell^*$ and $\ell \succ_h \ell^*$.*

Proof. $\ell^* = (\widehat{x}_{m-1}, 1-p; \widehat{x}_m, p)$. Lemma 1 (b) imlies that if $\ell \succsim_i (\widehat{x}_m, 1)$ and $\ell \succsim_i (\widehat{x}_{m-1}, 1)$, $i \in d, h$, the other player j has opposite strict preferences and $\ell \succ_j \ell^*$ cannot hold. So let $\ell \succsim_d (\widehat{x}_m, 1)$, $(\widehat{x}_{m-1}, 1) \succ_d \ell$ and $\ell \succsim_h (\widehat{x}_{m-1}, 1)$,$(\widehat{x}_m, 1) \succ_h \ell$. By lemma 1 (a), $\widehat{x}_{m-1} < g(\ell) < \widehat{x}_m$.

We construct ℓ' by increasing the weight p of $\widehat{x}_m$ by Δp (which benefits h) and increase the weight of $x < \widehat{x}_{m-1}$ by Δq such that $U^d(\ell') \geq U^d(\ell^*)$. Overall, $U^h(\ell') \geq U^h(\ell^*)$ needs to hold. In the appendix we show that ℓ' does not satify this condition. By concavity of u, increasing the weight of $x' > \widehat{x}_m$ instead of $\widehat{x}_m$ makes it more difficult to fulfill the conditions for h and d. Thus there exists no ℓ which h and d prefer to ℓ^*. ∎

In general, it is possible for two agents on opposing sides of the median to favor a lottery which the median voter rejects (Banks/Duggan, 2006). This occurs when the median voter prefers a policy with more extreme outcomes which incurs large losses on the other players due to convex costs, a case not covered by lemma 2.

By lemma 1 (c), there are incentives for successive deviations which ultimately bring about a partition where m wins:

Lemma 3 *For every $\pi \in \Pi$, either m runs as a singleton and wins or there exists π' where m wins and $\pi \ll \pi'$.*

Proof. If no agent runs, m joins the race to prevent x^0. Consider the stage where m and only agents on one side of $\widehat{x}_m$ are running. In that case m wants to run as a singleton and wins. Consider some stage k and π_k where S with $s_1 < \widehat{x}_m < s_J$ determines ℓ. By lemma 1 (c), there exists T_k with $\pi' \succ_{T_k} \pi_k$. Suppose T_k join D_k. At stage $k+1$ either m wants to run as a singleton and win or there is T_{k+1} who want to join D_{k+1}. Hence, $\pi_k \ll \pi'$. ■

Only partitions where m wins are farsightedly stable:

Theorem 1 *In every $\pi \in V$ in the unique farsighted stable set the party representing the median voter runs for election as a singleton and wins.*

Proof. Let Π_m the set of partitions where m runs and wins as a singleton. Say, $V = \Pi_m$. By lemma 3, V is externally stable, i.e. for all $\pi \notin \Pi_m$, there is $\pi' \in \Pi_m$ which blocks π via $\ll$. As all $\pi' \in \Pi_m$ are pay off equivalent, V is also internally stable. We still have to show, that there is no solution V' with $\widetilde{\pi} \in V'$, $\widetilde{\pi} \notin \Pi_m$. Suppose there is such V'. Because V is externally stable, $V \subset V'$ cannot hold. Hence, there exist $\pi' \in \Pi_m$, $\pi' \notin V'$. Let $\overrightarrow{S} = \{i \in N | \widehat{x}^i > \widehat{x}_m\}$ and $\underline{\Pi} \subset \Pi_m$ the set of partitions $\underline{\pi} \in \Pi_m$ where all $i \in \overrightarrow{S}$ have dropped out of the race. Similarly, $\overleftarrow{S} = \{i | \widehat{x}^i < \widehat{x}_m\}$ and $\overline{\Pi} \subset \Pi_m$ the set of partitions $\overline{\pi} \in \Pi_m$ where all $i \in \overleftarrow{S}$ have dropped out. Note that $(\overline{\Pi} \cup \underline{\Pi}) \nsubseteq V'$ because elements in the set block every $\pi \in \Pi \backslash \Pi_m$ (see proof of lemma 3). Recall $S^* = \{m-1, m\}$ and in π^* only S^* is running. Assume that $\pi^* \succsim_m \pi^{**}$ with $S^{**} = \{m, m+1\}$ only running in π^{**}. Thus π^* is blocked only by some $\pi' \in \Pi_m$: Deviations by either i with $\widehat{x}_i > \widehat{x}_m$ or j with $\widehat{x}_j < \widehat{x}_{m-1}$ alone do not change the outcome ℓ^*. Consider a sequence $\pi_k \xrightarrow{S_k} \pi_{k+1}$, $k = 0, .., K-1$, $\pi_K = \pi''$ with $\ell'' \neq \ell^*$. Say, the outcome is ℓ^* for all stages $k \leq \tau$. If for S_τ, $\pi'' \succ S_\tau \pi_\tau$, by lemma 2 $\pi'' \not\succ_{S_k} \pi_k$ for at least some S_k with $k < \tau$. Hence, $\pi^* \ll \pi''$ is not true.

First, suppose $\pi^* \notin V'$. The only element in Π_m which blocks π^* is $\underline{\pi}^* = \{\{m-1\}, \{m\}, D\}$ by deviation of m: Initial deviations in $\overrightarrow{S}$ frustrate m from deviating and agents in $\overleftarrow{S}$ do not want to move to $\pi \in \Pi_m$. Hence, by external stability, $\underline{\pi}^* \in V'$. Suppose some $\pi \in \underline{\Pi}$, $\pi \neq \underline{\pi}^*$ is blocked by π'. The initial deviation is by $T \subset \overrightarrow{S}$. But $\pi' \ll \underline{\pi}^*$ via the sequence: $\overleftarrow{S} \backslash \{m-1\}$ drop out (and $m-1$ wins); $\overrightarrow{S}$ drops out (and m wins). Hence, $\underline{\Pi} \subset V'$. Because $(\overline{\Pi} \cup \underline{\Pi}) \nsubseteq V'$, there must be $\overline{\pi}' \notin V'$. Suppose there is $\pi'' \in V'$ with $\overline{\pi}' \ll \pi''$ initiated by $T \subset \overleftarrow{S}$. But now there is a deviation by $T \subset \overrightarrow{S}$ and $\pi'' \ll \underline{\pi}''$. But $\underline{\pi}'' \in V'$, violating internal stability.

Next, suppose $\pi^* \in V'$. In that case, if $\pi \in \overline{\Pi}$, $\pi \notin V'$ because π is blocked by π^* through a sequence of deviations by $m-1$ and m. Moreover, $\underline{\pi}^* \notin V'$ because otherwise m deviates to $\underline{\pi}^*$. But then there must be $\pi'' \in V'$ with $\pi'' \gg \underline{\pi}^*$. Because the deviation must be initiated by agents in $\overrightarrow{S}$, $\pi'' \ll \pi^*$. Hence V' is not stable. This proves uniqueness of V. ■

The farsighted stable set can be obtained as an optimistic stable standard of behavior (Greenberg, 1990) for an appropriately defined situation, i.e. the "Harsanyi-Chwe"-situation (Xue, 1998). Any stable π is also stable with farsighted conservative agents. A partition where only $\{m\}$ runs is a Nash equilibrium. Lemma 1 (c) implies that it is also a strong Nash equilibrium: There is no coalition where all members want to enter.

The assumption of strictly concave preferences cannot be suspended. Suppose, in the example we had $u^i = -|x - \widehat{x}_i|$. In that case, 1, 2 and 3 running as singletons is farsightedly stable: 1 and 3 are indifferent between the median outcome and the lottery. Furthermore, the assumption that coalitions induce policy lotteries cannot be suspended. Suppose that a coalition directly implements $g(\ell)$ instead of ℓ. Then any coalition S with $g(\ell(S, \mathbf{p})) = \widehat{x}_m$ is also farsightedly stable. This event, however, occurs with probability zero.

References

[1] Anesi, V. (2006), Committees with Farsighted Voters: A New Interpretation of Stable Sets, Social Choice and Welfare 27: 595-610.

[2] Bandyopadhyay, S., Oak, M. (2004) Party Formation and Political Bargaining: A Model of Proportional Representation, FEEM Working Paper No. 98.2004.

[3] Banks, J.S., Duggan, J. (2006) A Social Choice Lemma on Voting over Lotteries with Applications to a Class of Dynamic Games. Social Choice and Welfare 26: 285-304.

[4] Besley, T., Coate, S. (1997), An Economic Model of Representative Democracy, Quarterly Journal of Economics 112: 85-114.

[5] Calvert, R. (1985), Robustness of the Multidimensional Voting Model: Candidate Motivation, Uncertainty, and Convergence, American Journal of Political Science 29: 69-95.

[6] Chwe, M.S. (1994), Farsighted Coalitional Stability, Journal of Economic Theory 63: 299-325.

[7] Cox, G.W. (1990), Centripetal and Centrifugal Incentives in Electoral Systems, American Journal of Political Science 34: 903-935.

[8] Downs, A. (1957), An Economic Theory of Democracy, Harper Collins.

[9] Greenberg, J. (1990) The Theory of Social Situations, Cambridge University Press.

[10] Hamlin, A., Hjortlund, M. (2000), Proportional Representation with Citizen Candidates. Public Choice 103: 2205-230.

[11] Harsanyi, J.C. (1974), An Equilibrium-Point Interpretation of Stable Sets and a Proposed Alternative Definition, Management Science 20: 1472-1492.

[12] Pech, G. (2009), Intra Party Decision Making, Party Formation and Moderation in Multiparty Democracy, mimeo.

[13] Shino, J. (2009), Farsighted Stable Sets in Hotelling's Location Games, working paper, Rutger's University.

[14] Xue, L. (1998) Coalitional Stability under Perfect Foresight, Economic Theory 11: 603-627.

4 Appendix: Proof of lemma 2

To complete the proof of lemma 2, let $\Delta p = p'(\widehat{x}_m) - p^*(\widehat{x}_m) > 0$ and $\Delta q = p'(x) > 0$. Note that $p'(\widehat{x}_{m-1}) = p^*(\widehat{x}_{m-1}) - \Delta p - \Delta q$. Say, $U^d(\ell') - U^d(\ell^*) \geq 0$, hence $\Delta q(u^d(x) - u^d(\widehat{x}_{m-1})) + \Delta p(u^d(\widehat{x}_m) - u^d(\widehat{x}_{m-1})) \geq 0$ where the first term is positive and the second negative. By concavity u, $\frac{\Delta q}{\Delta p} \geq \frac{-(u^d(\widehat{x}_m) - u^d(\widehat{x}_{m-1}))}{u^d(x) - u^d(\widehat{x}_{m-1})} > \frac{\widehat{x}_m - \widehat{x}_{m-1}}{\widehat{x}_{m-1} - x}$ for $u^d(x) - u^d(\widehat{x}_{m-1}) > 0$. Suppose that also $U^h(\ell') - U^h(\ell^*) \geq 0$, hence $\Delta q(u^h(x) - u^h(\widehat{x}_{m-1})) + \Delta p(u^h(\widehat{x}_m) - u^h(\widehat{x}_{m-1})) \geq 0$ with the first term negative and the second positive. By concavity of u, $\frac{\Delta q}{\Delta p} \leq \frac{u^h(\widehat{x}_m) - u^h(\widehat{x}_{m-1})}{-(u^h(x) - u^h(\widehat{x}_{m-1}))} < \frac{\widehat{x}_m - \widehat{x}_{m-1}}{\widehat{x}_{m-1} - x}$ which yields the desired contradiction.